THE PERIODIC 6-PARTICLE KAC-VAN MOERBEKE SYSTEM

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Abstract. We study some algebraic-geometrical aspects of the periodic 6-particle Kac-van Moerbeke system. This system is known to be algebraically integrable, having the affine part of a hyperelliptic Jacobian of a genus two curve as the generic fiber of its momentum map. Particular attention goes to the divisor needed to complete this fiber into an Abelian variety: it consists of six copies of the curve, intersecting according to a pattern which we will determine. We will also compare this divisor to the divisor which appears in some natural *singular* compactification of the fiber.

Contents

1. Introduction

The periodic n-particle Kac-van Moerbeke system (KM system) is given by the following quadratic vector field on $\mathbb{C}^n$:

$$\dot{x}_i = x_i(x_{i-1} - x_{i+1}) \;, \qquad i = 1, \dots, n \;, \tag{1.1}$$

where $x_0 := x_n$ and $x_{n+1} := x_1$. It is a Hamiltonian system with respect to the quadratic Poisson structure defined by

$$\{x_i, x_j\} := x_i x_j(\delta_{i,j+1} - \delta_{i+1,j}) \;, \qquad i, j = 1, \dots, n \;.$$

Indeed, taking $H := x_1 + x_2 + \cdots + x_n$ as Hamiltonian, the Hamiltonian vector field $\mathcal{X}_H := \{\cdot\,, H\}$ is precisely (1.1). The Poisson structure has rank $n-1$

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when n is odd and $n-2$ otherwise. The system was introduced by Kac and van Moerbeke in [6] who constructed this system as a discretization of the Korteweg-de Vries equation and who also showed its Liouville integrability. Several first integrals are produced from the Lax operator $L(\mathfrak{h})$, which is obtained from the Lax operator of the classical n-particle Toda lattice by replacing all diagonal elements by zero; see [5] for a precise account of this in terms of Poisson geometry. They yield $s := [(n+3)/2]$ independent first integrals, in involution, which is the exact number required to assure the Liouville integrability of the periodic n-particle KM system.

It was shown in [5] that all periodic KM systems are algebraically integrabile (a.c.i.). It means that for generic $\mathbf{c} := (c_1, \dots, c_s) \in \mathbb{C}^s$ the fiber $\mathbf{F_c} := \mathbf{F}^{-1}(\mathbf{c})$ of the momentum map (the map $\mathbf{F} : \mathbb{C}^n \to \mathbb{C}^s$, defined by the first integrals) is an affine part[1] of an Abelian variety $\mathbf{T_c}$ and that the integrable vector fields are translation invariant (with respect to the group structure on the Abelian varieties) on these fibers. In the present case of the periodic n-particle KM system, the Abelian variety $\mathbf{T_c}$ has two (equivalent) descriptions:

- As the Prym variety of the spectral curve $|\mathfrak{z}\mathrm{Id}_n - L_{\mathbf{c}}(\mathfrak{h})| = 0$, equipped with the involution $(\mathfrak{z}, \mathfrak{h}) \mapsto (-\mathfrak{z}, \mathfrak{h})$; here, $L_{\mathbf{c}}(\mathfrak{h})$ denotes the Lax operator $L(\mathfrak{h})$, restricted to $\mathbf{F_c}$;
- As hyperelliptic Jacobians, associated to the quotient of the above spectral curve by the involution $(\mathfrak{z}, \mathfrak{h}) \mapsto (-\mathfrak{z}, -\mathfrak{h})$.

Moreover, it is shown that the divisor which needs to be adjoined to the generic fiber $\mathbf{F_c}$ in order to complete it into $\mathbf{T_c}$ consists of n translates of the theta divisor.

In the case of $n = 5$ and of $n = 6$, the Abelian varieties are surfaces. For these cases an alternative proof of algebraic integrability can be given using the systematic method which was developed by Adler and van Moerbeke and presented in its final form in [2]. In fact, it is precisely the periodic 5-particle KM system which is used as a running example in [2] to present the method. Accessorily it provides an alternative proof of the algebraic integrability in the case $n = 5$. Similarly, such an alternative proof can be given in the case of $n = 6$. In this paper we will *not* present such an alternative proof, but study two compactifications of the fibers of the momentum map, using the known fact that the system is algebraically integrable. Notice however that our presentation will contain most elements, in particular all essential formulas, needed for providing the alternative proof which we just mentioned.

The two compactifications which we will consider of the generic fiber $\mathbf{F_c}$ are of a quite different character. The first one is the one which compactifies $\mathbf{F_c}$ into the torus $\mathbf{T_c}$. The six translates of the theta divisor (which

[1] When n is even, the fiber contains one or two isomorphic components, depending on the precise choice of momentum map $\mathbf{F}$; see Section 2 and in particular diagram (2.6) for details in the case of $n = 6$.

is in this case a curve, which can be identified with a quotient of the spectral curve) intersect each other in a quite particular, symmetric pattern: if we order the translates cyclically, every curve intersects its two neighboring curves in two different points, is tangent to its two second nearest neighbors and has two different intersection points with the remaining curve (its farest neighbor). The other compactification is the standard homogeneous compactification of the fiber, obtained by first compactifying $\mathbb{C}^6$ in the standard way to $\mathbb{P}^6$ and then taking the closure $\bar{\mathbf{F}}_{\mathbf{c}}$ of $\mathbf{F}_{\mathbf{c}}$ in $\mathbb{P}^6$. The resulting surface is singular: the divisor which has been added consists of 6 non-singular conics and three singular conics which are double lines and it is precisely along these lines that the surface $\bar{\mathbf{F}}_{\mathbf{c}}$ is singular. It will be clair that we rely heavily on the KM vector field for obtaining the first compactification, but not for the second one. Since the two compactifications are birationally isomorphic, it would be interesting to obtain the first compactification in a purely algebraic-geometrical way, i.e., without using the periodic 6-particle KM vector field. Since in this case the singularities are clearly identified and not too complicated, doing this may be feasible.

The plan of the paper is as follows. In Section 2 we recall the main results on the integrability and algebraic integrability of the periodic n-particle KM system and add some extra observations. An essential ingedient in the study of algebraic integrable systems is the family of Laurent solutions to the vector field(s) of the system. We give explicit formulas for (the first terms of) all Laurent solutions in Section 3. The principal balances (Laurent solution depending on 5 free parameters) are used in Section 4 to construct an embedding of the generic Abelian surfaces $\mathbf{T}_{\mathbf{c}}$ which compactify the generic fibers $\mathbf{F}_{\mathbf{c}}$ of the momentum map. The embedding allows us to compute an equation for the 6 (isomorphic) curves which make up the Painlevé divisor. This is done in Section 5, where we also relate the Painlevé curves to the spectral curve. In the final Section 6 we present the two compactifications of the generic surfaces $\mathbf{F}_{\mathbf{c}}$, with special attention to the geometry of the divisors which are adjoined in both cases.

2. The periodic 6-particle KM system

In this section we recall from [6] and [5] the main results on the Liouville, respectively algebraic integrability of the periodic n-particle Kac-van Moerbeke (KM) system, which we specialize to the case of $n = 6$. We also add a few extra observations which are specific to this case. The notions and notations which are introduced here will be used throughout the paper.

The periodic 6-particle KM system is given by the following quadratic vector field on $\mathbb{C}^6$:

$$\dot{x}_i = x_i(x_{i-1} - x_{i+1}) , \qquad i = 1, \dots, 6 . \tag{2.1}$$

Here, $x_1, \dots, x_6$ are the standard linear coordinates on $\mathbb{C}^6$; also, $x_7 = x_1$ and $x_0 = x_6$, i.e., all indices are taken modulo 6. The latter accounts for the adjective *periodic*. It is a Hamiltonian system with linear Hamiltonian

$$H := x_1 + x_2 + \cdots + x_6\ , \tag{2.2}$$

and with a quadratic Poisson structure, defined by the following formulas:

$$\{x_i, x_j\} := x_i x_j(\delta_{i,j+1} - \delta_{i+1,j})\ , \qquad i, j = 1, \dots, 6\ .$$

Some basic constants of motion of (2.1) are found by using a Lax operator. Indeed, (2.1) can be written as the following Lax equation with a spectral parameter (which we denote by $\mathfrak{h}$),

$$\dot{L}(\mathfrak{h}) = [L(\mathfrak{h}), M(\mathfrak{h})]\ ,$$

where $L(\mathfrak{h})$ and $M(\mathfrak{h})$ are given by

$$L(\mathfrak{h}) = \begin{pmatrix} 0 & x_1 & 0 & 0 & 0 & \mathfrak{h}^{-1} \\ 1 & 0 & x_2 & 0 & 0 & 0 \\ 0 & 1 & 0 & x_3 & 0 & 0 \\ 0 & 0 & 1 & 0 & x_4 & 0 \\ 0 & 0 & 0 & 1 & 0 & x_5 \\ \mathfrak{h}x_6 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{2.3}$$

and

$$M(\mathfrak{h}) = \begin{pmatrix} 0 & 0 & x_1x_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2x_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_3x_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_4x_5 \\ \mathfrak{h}x_5x_6 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathfrak{h}x_6x_1 & 0 & 0 & 0 & 0 \end{pmatrix}\ .$$

The characteristic polynomial of $L(\mathfrak{h})$ is given by

$$|\mathfrak{z}\mathrm{Id}_6 - L(\mathfrak{h})| = \mathfrak{z}^6 - F_1\,\mathfrak{z}^4 + F_2\,\mathfrak{z}^2 - \frac{1}{\mathfrak{h}}(1 + F_3\,\mathfrak{h})(1 + F_4\,\mathfrak{h})\ , \tag{2.4}$$

where the coefficients F_i are polynomial functions on $\mathbb{C}^6$; they are explicitly given by the following formulas:

$$\begin{aligned} F_1 &= x_1 + x_2 + x_3 + x_4 + x_5 + x_6\ , \\ F_2 &= x_1x_4 + x_2x_5 + x_3x_6 + x_1x_3 + x_2x_4 + x_3x_5 + x_4x_6 + x_1x_5 + x_2x_6\ , \\ F_3 &= x_1x_3x_5\ , \\ F_4 &= x_2x_4x_6\ . \end{aligned} \tag{2.5}$$

Notice that F_1 is just H, the Hamiltonian of the system. By a basic property of Lax equations, every coefficient (in $\mathfrak{z}$ and $\mathfrak{h}$) of (2.4) is a constant of motion of (2.1), hence the functions $F_1, \dots, F_4$ are constants of motion of (2.4). Both F_3 and F_4 are Casimir functions of the Poisson structure, whose rank is 4 at a generic point of $\mathbb{C}^6$ (to be precise: the rank is 4 at all points, except at those satisfying $x_i = x_{i+2} = 0$ for some $i \in \{1, \dots, 6\}$). Also, F_1 and F_2 are in involution since F_2 is a constant of motion of (2.1) and since $H = F_1$; it follows that the functions $F_1, \dots, F_4$ are pairwise in involution. Finally, they are also independent, so $(\mathbb{C}^6, \{\cdot\,,\cdot\}, (F_1,\ldots,F_4))$ defines a Liouville integrable system. We view $\mathbf{F} := (F_1, F_2, F_3, F_4)$ as a polynomial map $\mathbf{F} : \mathbb{C}^6 \to \mathbb{C}^4$, the *momentum map* of the integrable system. By what precedes, for a generic point $\mathbf{c} \in \mathbb{C}^4$ the fiber $\mathbf{F_c} := \mathbf{F}^{-1}(\mathbf{c})$ of $\mathbf{F}$ is a smooth complex surface to which the commuting Hamiltonian vector fields $\mathcal{X}_{F_1}$ and $\mathcal{X}_{F_2}$ are tangent; moreover, these vector fields generate the tangent space to the fiber $\mathbf{F_c}$ at each point. Finally, notice that the system is *homogeneous*, i.e., the Poisson structure $\{\cdot\,,\cdot\}$, the constants of motion $F_1,\ldots,F_4$ and the vector field $\mathcal{X}_H$ are weight homogeneous when all variables x_i are given weight 1 and time t is given weight -1.

We now turn to the algebraic integrability of the system, which yields a more precise description of the generic fiber $\mathbf{F_c}$ of $\mathbf{F}$. By construction, the characteristic polynomial of $L(\mathfrak{h})$ is constant on the fibers of $\mathbf{F}$ and we have the following commutative triangle:

$$\begin{tikzcd} & \mathbb{C}^6 \arrow[dl, "\mathbf{F}"'] \arrow[dr, "\mu"] & \\ \mathbb{C}^4 \arrow[rr, "2:1", "\rho"'] & & \mathcal{H}_6 \end{tikzcd} \tag{2.6}$$

In this diagram, $\mathcal{H}_6$ stands for the following space of Laurent polynomials:

$$\left\{ f_{\mathbf{c}}(\mathfrak{z},\mathfrak{h}) := \mathfrak{z}^6 - c_1\,\mathfrak{z}^4 + c_2\,\mathfrak{z}^2 - \frac{1}{\mathfrak{h}}(1+c_3\,\mathfrak{h})(1+c_4\,\mathfrak{h}) \mid \mathbf{c} = (c_1,\ldots,c_4) \in \mathbb{C}^4 \right\}$$

and ρ is defined for $\mathbf{c} \in \mathbb{C}^4$ by $\rho(\mathbf{c}) := f_{\mathbf{c}}(\mathfrak{z},\mathfrak{h})$; the definition of μ follows from it: $\mu := \rho \circ \mathbf{F}$. For $\mathbf{c} \in \mathbb{C}^4$ the Laurent polynomial $f_{\mathbf{c}}$ defines an algebraic curve, to wit the curve $f_{\mathbf{c}}(\mathfrak{z},\mathfrak{h}) = 0$, which is called the *spectral curve*. Setting

$$\nu := 2c_3c_4\mathfrak{h} - \mathfrak{z}^6 + c_1\mathfrak{z}^4 - c_2\mathfrak{z}^2 + c_3 + c_4\ ,$$

one easily computes that the spectral curve is birationally isomorphic to the affine algebraic curve $\Gamma_{\mathbf{c}}$, defined by equation $\nu^2 = g_{\mathbf{c}}(\mathfrak{z}^2)$, where

$$g_{\mathbf{c}}(\tau) := \tau(\tau^2 - c_1\tau + c_2)(\tau^3 - c_1\tau^2 + c_2\tau - 2(c_3+c_4)) + (c_3 - c_4)^2\ .$$

From this equation it is clear that $\Gamma_{\mathbf{c}}$ is for generic $\mathbf{c}$ a smooth hyperelliptic curve of genus $\left[\frac{2\deg g_{\mathbf{c}}-1}{2}\right] = \deg g_{\mathbf{c}} - 1 = 5$. Denoting the smooth compactification of $\Gamma_{\mathbf{c}}$ by $\bar{\Gamma}_{\mathbf{c}}$ we have a ramified double cover $\pi : \bar{\Gamma}_{\mathbf{c}} \to \mathbb{P}^1$. It is also clear from the equation of $\Gamma_{\mathbf{c}}$ that $\bar{\Gamma}_{\mathbf{c}}$ has three different involutions: first there is the hyperelliptic involution, defined on $\Gamma_{\mathbf{c}}$ by $\imath(\mathfrak{z},\nu) := (\mathfrak{z},-\nu)$; a second involution is defined by $\sigma(\mathfrak{z},\nu) := (-\mathfrak{z},\nu)$; since $\imath$ and σ commute, a third involution is defined by their composition, $\tau(\mathfrak{z},\nu) := (-\mathfrak{z},-\nu)$. Setting $\bar{\Gamma}^{\sigma}_{\mathbf{c}} := \bar{\Gamma}_{\mathbf{c}}/\sigma$ and $\bar{\Gamma}^{\tau}_{\mathbf{c}} := \bar{\Gamma}_{\mathbf{c}}/\tau$ the different curves can be represented by the following diagram:

$$\begin{tikzcd}
 & \bar{\Gamma}_{\mathbf{c}} \arrow[dl, "\pi_\tau"'] \arrow[dr, "\pi_\sigma"] \arrow[dd, "\pi"] & \\
\bar{\Gamma}_{\mathbf{c}}^{\tau} \arrow[dr] & & \bar{\Gamma}_{\mathbf{c}}^{\sigma} \arrow[dl] \\
 & \mathbb{P}^1 &
\end{tikzcd}$$

All maps in this diagram are double covers, with $12,\ 4, 0$ ramification points for $\pi,\ \pi_\tau$ and π_σ respectively. It follows that $\bar{\Gamma}_{\mathbf{c}}^{\tau}$ has genus 2 and that $\bar{\Gamma}_{\mathbf{c}}^{\sigma}$ has genus 3 (and that $\bar{\Gamma}_{\mathbf{c}}$ has genus 5, but we know that already). An explicit equation for an affine part of the quotient curves $\bar{\Gamma}_{\mathbf{c}}^{\tau}$ and $\bar{\Gamma}_{\mathbf{c}}^{\sigma}$ is respectively given by

$$\Gamma_{\mathbf{c}}^{\tau} : v^2 = g_{\mathbf{c}}(u)\ , \qquad \Gamma_{\mathbf{c}}^{\tau} : v^2 = u g_{\mathbf{c}}(u)\ .$$

Of course these quotient curves are also hyperelliptic, with their hyperelliptic involution $(u, v) \mapsto (u, -v)$ induced by $\imath$. The three involutions and the corresponding quotient curves play an important role in the description of the fibers of the momentum map $\mathbf{F}$ of the periodic 6-particle KM system:

Proposition 2.1 ([5])**.** *For generic $\mathbf{c} \in \mathbb{C}^4$, the fiber $\mathbf{F}_{\mathbf{c}}$ of the momentum map $\mathbf{F} = (F_1, \dots, F_4) : \mathbb{C}^6 \to \mathbb{C}^4$, with the F_i given by (2.5), is an affine part of*

$$\mathrm{Prym}(\bar{\Gamma}_{\mathbf{c}}/\Gamma_{\mathbf{c}}^{\sigma}) \simeq \mathrm{Jac}(\bar{\Gamma}_{\mathbf{c}}^{\tau})\ ,$$

obtained by removing 6 translates of the theta divisor. Moreover, the vector fields $\mathcal{X}_{F_1}$ and $\mathcal{X}_{F_2}$ are translation invariant on these tori.

We denote the divisor consisting of the 6 translates of the theta divisor by $\mathcal{D}_{\mathbf{c}}$ and we denote the complete Abelian surface, which we may think of as a Prym variety or as a hyperelliptic Jacobian, by $\mathbf{T}_{\mathbf{c}}$. Thus, $\mathbf{F}_{\mathbf{c}} = \mathbf{T}_{\mathbf{c}} \setminus \mathcal{D}_{\mathbf{c}}$.

The genericity condition on $\mathbf{c}$ in Proposition 2.1 can be made precise: the statement of the proposition holds precisely for those $\mathbf{c} \in \mathbb{C}^4$ for which the affine curve $\Gamma_{\mathbf{c}}$ is smooth. In what follows we will not need this precise description: we will only use that for generic $\mathbf{c} \in \mathbb{C}^4$ the curves $\Gamma_{\mathbf{c}}$, $\Gamma_{\mathbf{c}}^{\tau}$ and $\Gamma_{\mathbf{c}}^{\sigma}$ are smooth and that Proposition 2.1 holds for such $\mathbf{c}$. Also, in view of Diagram (2.6) and Proposition 2.1, the fibers of the momentum map μ over a generic Laurent polynomial $f_{\mathbf{c}} \in \mathcal{H}_6$ consists of the disjoint union of the isomorphic fibers $\mathbf{F}_{\mathbf{c}}$ and $\mathbf{F}_{\mathbf{c}'}$, where $\mathbf{c}'$ is obtained from $\mathbf{c}$ by permuting c_3 and c_4; thus, it is sufficient to study the fibers of $\mathbf{F}$ and we will not consider the fibers of μ in what follows.

3. Laurent solutions

In this section we determine all Laurent solutions of the periodic 6-particle KM vector field

$$\dot{x}_i = x_i(x_{i-1} - x_{i+1})\ , \qquad i = 1, \dots, 6\ , \tag{3.1}$$

where we recall that the indices are taken modulo 6 (so that for example $x_7 = x_1$ and $x_0 = x_6$). A *Laurent solution* to (3.1) is a 6-tuple of convergent (for small $t \neq 0$) Laurent series

$$x_i(t) = \frac{1}{t^{r_i}} \sum_{j=0}^{\infty} a_i^{(j)} t^j \ , \quad r_i \in \mathbb{Z} \ , \quad i = 1, \ldots, 6 \ ,$$

which yield a formal solution to (3.1). As we will see, the coefficients of these series depend polynomially on several parameters, where the parameter space is an affine variety which is not irreducible. The Laurent solutions to (3.1) are therefore naturally organized in *irreducible* families, i.e., families of Laurent solutions, parametrized by an irreducible affine variety. An irreducible family parametrized by an affine variety of dimension $n-1$ is called a *principal balance*; the other balances are called *lower balances*. According to the following theorem, known as the *Kowalevski-Painlevé Criterion*, every irreducible[2] a.c.i. system, such as the periodic 6-particle KM system, admits one or several principal balances.

Theorem 3.1 ([2, Theorem 6.13])**.** *Let* $(\mathbb{C}^n, \{\cdot\,,\cdot\}\,, \mathbf{F})$ *be an irreducible, polynomial a.c.i. system, where* $\mathbf{F} = (F_1, \ldots, F_s)$ *and let* $(x_1, \ldots, x_n)$ *be a system of linear coordinates on* $\mathbb{C}^n$*. Let* $\mathcal{X}$ *be any one of the integrable vector fields* $\mathcal{X}_{F_1}, \ldots, \mathcal{X}_{F_s}$*. For every* $1 \leqslant i \leqslant n$ *such that* x_i *is not constant along the integral curves of* $\mathcal{X}$ *there exists a principal balance* $x(t) = (x_1(t), \ldots, x_n(t))$ *for which* $x_i(t)$ *has a pole.*

Since the KM system is homogeneous, it is natural to look for *weight homogeneous* Laurent solutions of (3.1), i.e., Laurent solutions for which the pole order of $x_i(t)$ is at most the weight of the variable x_i, which is 1 for all variables x_i of the KM system (see Section 2; also, see [2, Section 7] for more information on (weight) homogeneous systems and Laurent solutions). We show in the following proposition by a simple argument that *all* Laurent solutions to the KM system are weight homogeneous.

Proposition 3.2. *Let*

$$x_i(t) = \frac{1}{t^{r}} \sum_{j=0}^{\infty} a_i^{(j)} t^j \ , \qquad i = 1, \ldots, 6 \ , \tag{3.2}$$

be a strict Laurent solution to the periodic n*-particle KM system, where* $a_i^{(0)} \neq 0$ *for at least one index* i*. Then* $r = 1$*.*

Proof. Suppose that a (3.2) is a *strict* Laurent solution, i.e., with $r \geqslant 1$ (otherwise it would be a Taylor solution). Notice that not all $a_i^{(0)}$ with i odd can be different from zero because the product $x_1 x_3 x_5$ is a constant of

[2]An a.c.i. system is said to be *irreducible* if for generic $\mathbf{c}$ the Abelian variety compactifying the fiber $\mathbf{F_c}$ of its momentum map $\mathbf{F}$ is a *simple* Abelian variety, i.e., it contains no proper Abelian subvarieties.

motion of (3.2), and so $x_1(t)x_3(t)x_5(t)$ is independent of t. Similarly, not all $a_i^{(0)}$ with i even can be different from zero. It follows that there exists an index i such that exactly one of $a_{i-1}^{(0)}$ and $a_{i+1}^{(0)}$ has a pole order r (and so the pole order of the other one is smaller). Since $x_i(t)$ is not identically zero (x_i is not a constant of motion), we may consider $\dot{x}_i(t)/x_i(t)$ which has at most a simple pole, but is in view of (3.1) equal to $x_{i-1}(t) - x_{i+1}(t)$, which has pole of order r. Hence $r = 1$ as was to be shown. $\square$

It follows that all Laurent solutions to (3.1) are weight homogeneous, can be algorithmically computed (see [2, Proposition 7.6]) and are convergent (see [2, Theorem 7.25]). Setting

$$x_i(t) = \frac{1}{t}\sum_{j=0}^{\infty} a_i^{(j)} t^j \ , \qquad i = 1, \dots, 6 \ , \tag{3.3}$$

one first solves the *indicial equations* which are the non-linear equations obtained by substituting the Laurent solutions (3.3) in (3.1) and equating the lowest order terms, i.e., the terms in t^{-2}. The result is the following system of quadratic equations:

$$-a_i^{(0)} = a_i^{(0)} \left(a_{i-1}^{(0)} - a_{i+1}^{(0)}\right) \ , \qquad i = 1, \dots, 6 \ . \tag{3.4}$$

All non-zero solutions of (3.4) are easily found as follows. First recall from the proof of Proposition 3.2 that $a_i^{(0)} = 0$ for at least one odd and for at least one even value of i. We may therefore assume (by a cyclic permutation of the indices, if needed) that $a_1^{(0)} \neq 0$ and that $a_6^{(0)} = 0$. Then $a_3^{(0)} = 0$ or $a_5^{(0)} = 0$. When $a_3^{(0)} = 0$, the remaining equations in (3.4) lead to $a_1^{(0)} = -a_2^{(0)} = -1$, and

$$(a_5^{(0)} = 1 \text{ or } a_4^{(0)} = 0) \quad \text{and} \quad (a_4^{(0)} = -1 \text{ or } a_5^{(0)} = 0) \ ,$$

which leads to two solutions with $a_1^{(0)} \neq 0$ and $a_6^{(0)} = 0$, to wit $a^{(0)} = (-1, 1, 0, -1, 1, 0)$ and $a^{(0)} = (-1, 1, 0, 0, 0, 0)$. Similarly, when $a_5^{(0)} = 0$ we find a single new solution $a^{(0)} = (-2, 1, -1, 2, 0, 0)$. The upshot is that the indicial equations have 15 non-trivial solutions, to wit $(-1, 1, 0, 0, 0, 0)$, $(-1, 1, 0, -1, 1, 0)$, $(-2, 1, -1, 2, 0, 0)$ and their cyclic permutations (notice that the second solution has only three *different* cyclic permutations).

Having determined all possibilities for the leading coefficients of the Laurent series $x_i(t)$, we need to investigate the existence of the subsequent terms in the series as well as their dependence on free parameters. This has to be done seperately for each solution $a^{(0)}$ to the indicial equations. Thanks to the order 6 symmetry, we only need to consider the above three particular solutions. In each case, the subsequent terms $a_i^{(k)}$ are for $k = 1, 2, 3, \dots$ determined by the following linear problem:

$$\left(k\mathrm{Id}_6 - \mathcal{K}(a^{(0)})\right) a^{(k)} = R^{(k)} \ , \tag{3.5}$$

where $\mathcal{K}(a^{(0)})$ is the Kowalevski matrix, evaluated at $a^{(0)}$ and $R^{(k)}$ is a column vector of polynomials which depends on the coefficients $a_1^{(j)}, \dots, a_6^{(j)}$ with $0 \leqslant j < k$ only. Thanks to homogeneity, the Kowalevski matrix is just the sum of the Jacobian matrix of the right hand side of (3.1), evaluated at $a^{(0)}$, plus the identity matrix (see [2, Proposition 7.6] for the formula for the Kowalevski matrix in the general weight homogeneous case), namely $\mathcal{K}(a^{(0)})$ is given by

$$\begin{pmatrix} a_6^{(0)} - a_2^{(0)} + 1 & -a_1^{(0)} & 0 & 0 & 0 & a_1^{(0)} \\ a_2^{(0)} & a_1^{(0)} - a_3^{(0)} + 1 & -a_2^{(0)} & 0 & 0 & 0 \\ 0 & a_3^{(0)} & a_2^{(0)} - a_4^{(0)} + 1 & -a_3^{(0)} & 0 & 0 \\ 0 & 0 & a_4^{(0)} & a_3^{(0)} - a_5^{(0)} + 1 & -a_4^{(0)} & 0 \\ 0 & 0 & 0 & a_5^{(0)} & a_4^{(0)} - a_6^{(0)} + 1 & -a_5^{(0)} \\ -a_6^{(0)} & 0 & 0 & 0 & a_6^{(0)} & a_5^{(0)} - a_1^{(0)} + 1 \end{pmatrix}.$$

For most values of k, namely for those which do not belong to the spectrum of $\mathcal{K}(a^{(0)})$ the linear equation (3.5) has a unique solution. When k is an eigenvalue of $\mathcal{K}(a^{(0)})$ of multiplicity μ_k, we can get at most μ_k free parameters at step k; in fact, it may happen that for the solvability of (3.5) one needs to impose conditions on $R^{(k)}$, i.e. on the free parameters which have been introduced in the previous steps, or it may even happen that (3.5) has no solution at all, independently of the values of those free parameters, which means that there is no Laurent solution with $a^{(0)}$ as leading coefficients. We will see that in the present case, the number of free parameters at each step k is equal to the multiplicity of k as an eigenvalue of $\mathcal{K}(a^{(0)})$, for any of the values of $a^{(0)}$ that we have found.

In order to do this, we first compute the characteristic polynomial of $\mathcal{K}(a^{(0)})$ for the above three particular values of $a^{(0)}$. To start with, consider

$$\mathcal{K}(-1,1,0,0,0,0) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

Thanks to the almost upper triangular form of this matrix, we obtain at once the following formula for its characteristic polynomial:

$$\chi(\mathcal{K}(-1,1,0,0,0,0),\lambda) = (\lambda+1)(\lambda-1)^3(\lambda-2)^2 .$$

Similarly, one obtains

$$\begin{aligned} \chi(\mathcal{K}(-2,1,-1,2,0,0),\lambda) &= (\lambda+2)(\lambda+1)(\lambda-1)(\lambda-2)(\lambda-3)^2 , \\ \chi(\mathcal{K}(-1,1,0,-1,1,0),\lambda) &= (\lambda+1)^2(\lambda-1)^2(\lambda-3)^2 . \end{aligned}$$

In the first case we have 5 positive eigenvalues, while there are only 4 positive eigenvalues in the two other cases. So only the first case can lead to principal balances; however we know that the system is a.c.i., hence it must have principal balances (Theorem 3.1), and so $a^{(0)} = (-1,1,0,0,0,0)$ leads —

just like its cyclic permutations — to a principal balance. We exhibit the first few terms of it, for future use:

$$\begin{aligned}
x_1(t) &= -\frac{1}{t}+a-\frac{1}{3}(a^2-2d+e)t-\frac{1}{8}(8ad-be-3cd)t^2+\mathcal{O}(t^3)\ ,\\
x_2(t) &= \frac{1}{t}+a+\frac{1}{3}(a^2+d-2e)t-\frac{1}{8}(8ae-3be-cd)t^2+\mathcal{O}(t^3)\ ,\\
x_3(t) &= et+e(a-b)t^2+\mathcal{O}(t^3)\ ,\\
x_4(t) &= b-bct-\frac{b}{2}(bc-c^2-e)t^2+\mathcal{O}(t^3)\ ,\\
x_5(t) &= c+bct+\frac{c}{2}(b^2-bc+d)t^2+\mathcal{O}(t^3)\ ,\\
x_6(t) &= -dt+d(a-c)t^2+\mathcal{O}(t^3)\ .
\end{aligned} \tag{3.6}$$

In these formulas, $a, b, \dots, e$ are the five free parameters; a, b and c appear at the first step, while c and d appear at the second step, in agreement with the eigenvalues of the Kowalevski matrix. The subsequent terms are completely determined by the displayed terms because 2 is the largest eigenvalue.

For $a^{(0)} = (-2, 1, -1, 2, 0, 0)$ we get a lower balance, depending on 4 free parameters a, b, c, d, appearing at steps 1, 2 and 3. For future use, we also give its first few terms:

$$\begin{aligned}
x_1(t) &= -\frac{2}{t}+2a-2bt-2ct^2+\mathcal{O}(t^3)\ ,\\
x_2(t) &= \frac{1}{t}+a+(a^2-2b)t+(a^3-3ab-3c+d)t^2+\mathcal{O}(t^3)\ ,\\
x_3(t) &= -\frac{1}{t}+a-(a^2-2b)t+(3ab-a^3+7c-3d)t^2+\mathcal{O}(t^3)\ ,\\
x_4(t) &= \frac{2}{t}+2a+2bt+(12ab-4a^3+18c-8d)t^2+\mathcal{O}(t^3)\ ,\\
x_5(t) &= (4a^3-12ab-20c+9d)t^2+\mathcal{O}(t^3)\ ,\\
x_6(t) &= dt^2+\mathcal{O}(t^3)\ .
\end{aligned} \tag{3.7}$$

Finally, for $a^{(0)} = (-1, 1, 0, -1, 1, 0)$ we also get a lower balance, depending on 4 free parameters a, b, c, d, which appear at steps 1 and 3. Its first few terms are given by

$$\begin{aligned}
x_1(t) &= -\frac{1}{t}-b-\frac{b^2}{3}t+(c-3d)t^2+\mathcal{O}(t^3)\ ,\\
x_2(t) &= \frac{1}{t}-b+\frac{b^2}{3}t-dt^2+\mathcal{O}(t^3)\ ,\\
x_3(t) &= ct^2+\mathcal{O}(t^3)\ ,
\end{aligned}$$

$$
\begin{aligned}
x_4(t) &= -\frac{1}{t} + a - \frac{a^2}{3}t - dt^2 + \mathcal{O}(t^3) \ , \qquad (3.8)\\
x_5(t) &= \frac{1}{t} + a + \frac{a^2}{3}t + (c-3d)t^2 + \mathcal{O}(t^3) \ ,\\
x_6(t) &= (8d-3c)t^2 + \mathcal{O}(t^3) \ .
\end{aligned}
$$

4. Embedding the Abelian surfaces

We now construct a projective embedding of the Abelian surfaces $\mathbf{T_c}$ which compactify the generic fiber $\mathbf{F_c}$ of the momentum map of the periodic 6-particle KM system. To do this, we use the methods developed in [2], which we recall here and which we adapt to this system; there are some simplifications due to the fact that we know already that the latter system is an irreducible a.c.i. system.

First, recall that by definition every complex Abelian variety embeds in projective space and that such an embedding can be constructed by using the sections of a very ample line bundle on it; in the case of Abelian variety the third power of any ample line bundle suffices. In the present case, the generic fiber $\mathbf{F_c}$ of the momentum map is an affine part of a hyperelliptic Jacobian and the divisor to be adjoined to $\mathbf{F_c}$ to complete it into the torus $\mathbf{T_c}$ consists of 6 translates of the theta divisor, so it is very ample. We will therefore look for a basis of the sections of the line bundle defined by the divisor at infinity. Said differently, we look for meromorphic functions on the fiber, having at most a simple pole along the divisor at infinity. According to [2, Proposition 6.14] this can be done using the Laurent solutions: if we denote in the present case by $x(t;\mathcal{D}^i)$ the family of Laurent solutions corresponding to a Painlevé wall[3] $\mathcal{D}^i$ and f is a rational function of $x_1,\dots,x_6$ then the pole order (in t) of $f(x(t;\mathcal{D}^i))$ equals, for generic $\mathbf{c}\in\mathbb{C}^4$, the pole order of f, viewed as a meromorphic function on $\mathbf{T_c}$, along $\mathcal{D}^i_{\mathbf{c}}$. For example, it suffices to look at the pole orders of the Laurent series (3.6) to determine the divisor of zeros and poles of the coordinate functions $x_1,\dots,x_6$, restricted to the generic Abelian surface $\mathbf{T_c}$. The result is displayed in Table 1.

In it, the labelings of the Painlevé walls $\mathcal{D}^i$ are chosen such that $x(t;\mathcal{D}^1)$ is the principal balance (3.6) and the other labelings are obtained by a cyclic permutation of the variables, i.e., $x_i(t,\mathcal{D}^2) = x_{i-1}(t,\mathcal{D}^1)$, and so on. In a single formula, the table can be summarized by

$$
(x_i)|_{\mathbf{T_c}} = \mathcal{D}^{i+1}_{\mathbf{c}} - \mathcal{D}^{i}_{\mathbf{c}} - \mathcal{D}^{i-1}_{\mathbf{c}} + \mathcal{D}^{i-2}_{\mathbf{c}} \ .
$$

From this formula we can for example conclude that the divisors $\mathcal{D}^i_{\mathbf{c}}+\mathcal{D}^{i-1}_{\mathbf{c}}$ and $\mathcal{D}^{i+1}_{\mathbf{c}}+\mathcal{D}^{i-2}_{\mathbf{c}}$ are linearly equivalent for all i.

[3] Roughly speaking, the Painlevé wall $\mathcal{D}^i$ is the collection of Painlevé divisors $\mathcal{D}^i_{\mathbf{c}}$, with $\mathbf{c}\in\mathbb{C}^4$ generic; see [2, Chapter 6] for a precise description of $\mathcal{D}^i$ as a divisor on a partial compactification of phase space.

	$\mathcal{D}^1_{\mathbf{c}}$	$\mathcal{D}^2_{\mathbf{c}}$	$\mathcal{D}^3_{\mathbf{c}}$	$\mathcal{D}^4_{\mathbf{c}}$	$\mathcal{D}^5_{\mathbf{c}}$	$\mathcal{D}^6_{\mathbf{c}}$
x_1	-1	1	0	0	1	-1
x_2	-1	-1	1	0	0	1
x_3	1	-1	-1	1	0	0
x_4	0	1	-1	-1	1	0
x_5	0	0	1	-1	-1	1
x_6	1	0	0	1	-1	-1

TABLE 1. The divisor of zeros and poles of the coordinate functions $x_1, \dots, x_6$, restricted to the generic Abelian surface $\mathbf{T_c}$.

k	$\dim \mathcal{F}^{(k)}$	$\dim \mathcal{H}^{(k)}$	$\dim \mathcal{Z}^{(k)}$	# dep	ζ_k	indep. functions
0	1	1	1	0	1	z_0
1	6	1	6	1	5	$z_1, \dots, z_5$
2	21	2	15	7	8	$z_6, \dots, z_{13}$
3	56	4	32	22	10	$z_{14}, \dots, z_{23}$
4	126	5	57	51	6	$z_{24}, \dots, z_{29}$
5	252	7	96	90	6	$z_{30}, \dots, z_{35}$
6	462	11	144	144	0	—

TABLE 2. Computing a basis for the polynomials of degree at most 6 which have a simple pole at most when any principal balance $x(t)$ is substituted in them.

Of course, the coordinate functions $x_1, \dots, x_6$ and the constant function 1 are the first elements of the polynomial functions we are looking for. In order to find the other such polynomials f, one proceeds by the degree d of f, looking for the most general polynomial of degree d such that $f(x(t; \mathcal{D}^i))$ has at most a simple pole (in t) for all i. Notice that we can take f to be homogeneous, because when f has the desired property, then by homogeneity every homogeneous component of f will also have this property. A more delicate issue is that we will find some polynomials which are dependent on the previously found polynomials, when restricted to the tori; in fact, if one multiplies any polynomial with the desired property with a constant of motion, the product will still have the desired property, without leading to a new meromorphic function, when restricted to the tori. The results of the process are summarized in Table 2: In the table are displayed, for small k, the following data, corresponding to the different columns (in that order).

(1) $\dim \mathcal{F}^{(k)}$, the number of linearly independent monomials of degree k; it is computed from the formula $\dim \mathcal{F}^{(k)} = \binom{k+5}{5}$;
(2) $\dim \mathcal{H}^{(k)}$, the number of linearly independent constants of motion of degree k; it is the coefficient in t^k of $((1-t)(1-t^2)(1-t^3)^2)^{-1}$;
(3) $\dim \mathcal{Z}^{(k)}$, the number of linearly independent polynomials having a simple pole at most when the principal balances are substituted in them; for computing this, a computer program is very useful;
(4) The number of linearly independent elements in $\mathcal{Z}^{(k)}$ that are dependent of the previous ones over $\mathcal{H}$. This is number is computed from the previous data by the formula $\sum_{j=0}^{i-1} \zeta_j \dim \mathcal{H}^{(i-j)}$;
(5) ζ_k, the number of linearly independent elements in $\mathcal{Z}^{(k)}$ that are independent of the previous ones over $\mathcal{H}$; it is computed as the difference of the two previous columns;
(6) The last column gives a choice of these new functions; their explicit expressions are given below.

We now list the functions z_i and explain why we need not look at polynomials of higher degree. In degree zero we have the constant function $z_0 := 1$; in degree one all coordinate functions $x_1, \ldots, x_6$ have a simple pole at most when the principal balances are substituted in them, but they are not independent over the Hamiltonians, since their sum is the Hamiltonian H_1. So we set $z_i := x_i$ for $i = 1, \ldots, 5$. The 8 quadratic polynomials are given by

$$\begin{aligned} &z_6 := x_1x_4 \ , \quad z_7 := x_2x_5 \ , \quad z_8 := x_3x_6 \ , \quad z_9 := x_1x_3 \ , \\ &z_{10} := x_2x_4 \ , \quad z_{11} := x_3x_5 \ , \quad z_{12} := x_4x_6 \ , \quad z_{13} := x_1x_5 \ , \end{aligned}$$

while the 10 cubic polynomials are given by

$$\begin{aligned} &z_{14} := x_1x_2x_3 \ , && z_{15} := x_2x_3x_4 \ , && z_{16} := x_3x_4x_5 \ , \\ &z_{17} := x_4x_5x_6 \ , && z_{18} := x_1x_5x_6 \ , && z_{19} := x_1x_2x_6 \ , \\ &z_{20} := x_1x_3(x_1+x_6) \ , && z_{21} := x_2x_4(x_1+x_2) \ , \\ &z_{22} := x_3x_5(x_3+x_2) \ , && z_{23} := x_4x_6(x_4+x_3) \ . \end{aligned}$$

Next follow the 6 quartic polynomials,

$$\begin{aligned} &z_{24} := x_1x_2x_3x_4 \ , \quad z_{25} := x_2x_3x_4x_5 \ , \quad z_{26} := x_3x_4x_5x_6 \ , \\ &z_{27} := x_1x_4x_5x_6 \ , \quad z_{28} := x_1x_2x_5x_6 \ , \quad z_{29} := x_1x_2x_3x_6 \ , \end{aligned}$$

and the 6 quintic polynomials

$$\begin{aligned} &z_{30} := x_1^2x_2x_3^2 \ , \quad z_{31} := x_2^2x_3x_4^2 \ , \quad z_{32} := x_3^2x_4x_5^2 \ , \\ &z_{33} := x_4^2x_5x_6^2 \ , \quad z_{34} := x_5^2x_6x_1^2 \ , \quad z_{35} := x_1x_2x_3x_4(x_1+x_2) \ . \end{aligned}$$

Notice that these polynomials are all either monomials or binomials, which makes it very easy to determine their leading behaviour; in fact, for the 30 monomials it suffices to look at Table 1 to verify that along any of the six curves $\mathcal{D}_{\mathbf{c}}^i$ their pole order is at most one!

On the generic torus $\mathbf{T}_{\mathbf{c}}$, which is a Jacobian surface, we have 36 independent functions with a simple pole at most along the divisor at infinity. Since this divisors consists of 6 translates of the theta divisor, it defines a polarization of type $(6,6)$ on the Abelian surface, and so $\dim H^0(\mathbf{T_c},\mathcal{D}_\mathbf{c}) = 6^2 = 36$. Therefore, we have constructed a basis of this space and the given function provide an embedding of the generic torus $\mathbf{T_c}$ in $\mathbb{P}^{35}$. We will use this embedding to determine the intersection pattern of these 6 theta translates.

5. The spectral and Painlevé curves

Recall that for generic $\mathbf{c} \in \mathbb{C}^4$ we denote by $\mathbf{F_c}$ the fiber $\mathbf{F}^{-1}(c)$ of the momentum map $\mathbf{F} : \mathbb{C}^6 \to \mathbb{C}^4$, by $\mathbf{T_c}$ its completion into an Abelian surface and by $\mathcal{D}_\mathbf{c}$ the divisor of $\mathbf{T_c}$ which has been added to do this completion. As we have seen, the divisor $\mathcal{D}_\mathbf{c}$ has six irreducible components, which are the Painlevé curves $\mathcal{D}^i_\mathbf{c}$, $i = 1,\dots,6$. These 6 curves are isomorphic, so in order to compute an affine equation for the Painlevé curves, it suffices to compute the equation for one of them; yet, as we will see in the next section, some care has to be taken when considering the 6 different projective embeddings of these curves, rather than the isomorphism class which they define.

In order to compute an affine equation for one of the $\mathcal{D}^i_\mathbf{c}$, one fixes $\mathbf{c} = (c_1,\dots,c_4)$ and substitutes the principal balance (3.6) of (3.1) in the equations $H_j = c_j$, for $j = 1,\dots,4$. Since H_j is a constant of motion, $H_j(x(t))$ is independent of t, hence depends on the free parameters $a,\dots,e$ only. We therefore get 4 polynomial equations in these parameters, and they give equations for an affine part of any one of the Painlevé curves. Notice that due to the simple form of the Hamiltonians, we only need to substitute the first two terms of the Laurent series (3.6) in $H_1 = c_1$, the first three terms in $H_2 = c_2$ and the leading terms in $H_3 = c_3$ and in $H_4 = c_4$. One obtains the following equations:

$$\begin{aligned} &2a+b+c = c_1 \ , \\ &2a(b+c) - d - e = c_2 \ , \\ &-ce = c_3 \ , \\ &-bd = c_4 \ . \end{aligned} \tag{5.1}$$

This curve is called the *(abstract) Painlevé curve*; we denote it by $\Delta_\mathbf{c}$. Solving the first and last two equations linearly for a, b and c and substituting the results in the second equation yields, after clearing the denominator, the following equation for a curve, birationally equivalent to $\Delta_\mathbf{c}$:

$$d^2e^2(e+d+c_2) + (c_3d+c_4e)(c_1de+c_3d+c_4e) = 0 \ . \tag{5.2}$$

We show that this curve is also birationally equivalent to the curve $\Gamma^\tau_\mathbf{c}$, which we constructed as a quotient of the spectral curve $\Gamma_\mathbf{c}$. Recall from Section 2 that an equation for $\Gamma^\tau_\mathbf{c}$ is given by

$$v^2 = u(u^2 - c_1u + c_2)(u^3 - c_1u^2 + c_2u - 2(c_3+c_4)) + (c_3-c_4)^2 \ . \tag{5.3}$$

The birational map is given by

$$u=-\frac{c_3}{e}-\frac{c_4}{d}\;,v=-\frac{d^2e(d+e+c_2)+c_1d(c_3d+c_4e)+c_4(2c_3d+(c_3+c_4)e)}{dc_3}\;,$$

with inverse map

$$d,e=-\frac{1}{2}\left(u^2-c_1u+c_2\pm\frac{v-c_3+c_4}{u}\right)\;,$$

where the plus sign corresponds to d and the minus sign to e. These formulas are easily checked by direct computation; to find the above map, one can for example write (5.2) in Weierstraß form and then rescale the variables so as to make the equation match with the Weierstraß form (5.3) of $\Gamma^\tau_{\mathbf{c}}$. The fact that the compactified Painlevé curve corresponding to $\mathbf{c}$ is isomorphic to $\bar\Gamma^\tau_{\mathbf{c}}$ is not surprizing since on the one hand the Painlevé curve is a divisor of the torus $\mathbf{T_c}$ and on the other hand $\mathbf{T_c}$ is isomorphic to the Jacobian of $\bar\Gamma^\tau_{\mathbf{c}}$.

We will need in the next section the points at infinity of (5.1), i.e., the points needed to complete the affine curve defined by (5.1) into a compact Riemann surface. We will in fact need a local parametrization around each of these points. It is important that we do this with the representation of the curve in terms of the parameters which appear in the Laurent series, rather than using some (possibly simpler) birational model, such as (5.2), because the embedding of $\mathbf{T_c}$ was constructed by using the Laurent solutions, and so the corresponding embeddings of the curve which we will construct will also be expressed in terms of these parameters. In order to find these parametrizations, it suffices to first observe that $bcde\neq 0$ for any affine point (recall that $\mathbf{c}$ is generic), so that for the points at infinity at least one of the parameters b,c,d,e must be zero; also notice that b and d cannot vanish at the same time, and similarly for c and e. In fact, out of c and e exactly one has to vanish, and similarly for b and d. For each of the four possibilities we find a single parametrization, except when d and e vanish, in which case we find two parametrizations. Thus we have five points at infinity. Local parametrizations around these points are given by the following list (we only give the parametrization for two of the variables; one easily derives from them parametrizations for the other variables by using (5.1)):

$$\begin{array}{lll}
\infty_1: & e=-c_3\tau\;, & d=c_4\tau(1+\beta\tau)+\mathcal{O}(\tau^3)\;,\\
\infty_2: & e=-c_3\tau\;, & d=c_4\tau(1+(c_1-\beta)\tau)+\mathcal{O}(\tau^3)\;,\\
\infty_3: & b=c_4\tau\;, & c=-c_3\tau(1+c_2\tau+(c_1c_3-c_1c_4+c_2^2)\tau^2)+\mathcal{O}(\tau^4)\;,\\
\infty_4: & d=c_4\tau\;, & c=c_3\tau^2(1-c_1\tau)+\mathcal{O}(\tau^4)\;,\\
\infty_5: & e=c_3\tau\;, & b=c_4\tau^2(1-c_1\tau)+\mathcal{O}(\tau^4)\;.
\end{array}$$

In the first two formulas, β stands for the same root of the quadratic polynomial $\beta^2-c_1\beta+c_2$; picking the other root just amounts to permuting the two points ∞_1 and ∞_2.

It is also useful to restrict the lower balances to the generic fibers $\mathbf{F_c}$. When doing so, one gets explicit formulas for the four free parameters in terms of the 4 constants $c_1, \ldots, c_4$, so that the Laurent solutions can be entirely expressed in terms of the latter constants. For the lower balances (3.7), the resulting Laurent solutions are given by

$$\begin{aligned}
x_1(t) &= -\frac{2}{t} + \frac{c_1}{3} - \frac{1}{18}(c_1^2 - 3c_2)t + \frac{1}{540}(2c_1^3 - 9c_1c_2 + 27c_3 - 243c_4)t^2 + \mathcal{O}(t^3)\ ,\\
x_2(t) &= \frac{1}{t} + \frac{c_1}{6} - \frac{1}{36}(c_1^2 - 6c_2)t - \frac{1}{1080}(4c_1^3 - 18c_1c_2 - 81c_3 + 189c_4)t^2 + \mathcal{O}(t^3)\ ,\\
x_3(t) &= -\frac{1}{t} + \frac{c_1}{6} + \frac{1}{36}(c_1^2 - 6c_2)t - \frac{1}{1080}(4c_1^3 - 18c_1c_2 + 189c_3 - 81c_4)t^2 + \mathcal{O}(t^3)\ ,\\
x_4(t) &= \frac{2}{t} + \frac{c_1}{3} + \frac{1}{18}(c_1^2 - 3c_2)t + \frac{1}{540}(2c_1^3 - 9c_1c_2 - 243c_3 + 27c_4)t^2 + \mathcal{O}(t^3)\ ,\\
x_5(t) &= \frac{c_3}{2}t^2 + \mathcal{O}(t^3)\ ,\\
x_6(t) &= \frac{c_4}{2}t^2 + \mathcal{O}(t^3)\ ,
\end{aligned}$$

while for the lower balances (3.8) they are given by

$$\begin{aligned}
x_1(t) &= -\frac{1}{t} + \frac{\beta}{2} - \frac{\beta^2}{12}t + \frac{1}{8}(3c_3 + c_4)t^2 + \mathcal{O}(t^3)\ ,\\
x_2(t) &= \frac{1}{t} + \frac{\beta}{2} + \frac{\beta^2}{12}t + \frac{1}{8}(c_3 + 3c_4)t^2 + \mathcal{O}(t^3)\ ,\\
x_3(t) &= -c_4t^2 + \mathcal{O}(t^3)\ ,\\
x_4(t) &= -\frac{1}{t} - \frac{1}{2}(\beta - c_1) - \frac{1}{12}(\beta - c_1)^2t + \frac{1}{8}(c_3 + 3c_4)t^2 + \mathcal{O}(t^3)\ ,\\
x_5(t) &= \frac{1}{t} - \frac{1}{2}(\beta - c_1) + \frac{1}{12}(\beta - c_1)^2t + \frac{1}{8}(3c_3 + c_4)t^2 + \mathcal{O}(t^3)\ ,\\
x_6(t) &= -c_3t^2 + \mathcal{O}(t^3)\ ,
\end{aligned}$$

where β is any root of he quadratic polynomial $\beta^2 - c_1\beta + c_2$. Notice that this means that, restricted to the generic fiber $\mathbf{F_c}$ we do not have just three but six of the latter lower balances. This will be reflected in the geometry of the divisor at infinity.

In order to obtain from these formulas the formulas for all lower balances one uses the order six automorphism, but one should not forget that it permutes also the constants c_3 and c_4.

6. The configuration of Painlevé curves

In this section we use the embedding of the Abelian surfaces $\mathbf{T_c}$ in $\mathbb{P}^{35}$ to construct six projective embeddings of the smooth Painlevé curve $\Delta_{\mathbf{c}}$, which we recall is birationally isomorphic to the smooth genus 2 curve $\Gamma_{\mathbf{c}}^{\tau}$. We will then be able to determine the intersection pattern of the 6 completed image curves which make up the Painlevé divisor $\mathcal{D}_{\mathbf{c}}$ of $\mathbf{T_c}$. To do this, we first substitute the principal balance (3.6) in the embedding functions $z_0, \ldots, z_{35}$, constructed in Section 4, which gives an embedding of a neighborhood in $\mathbf{T_c}$ of an affine part of the embedded curve, times a neighborhood of 0, corresponding to time t (the parameter of the integral curves of the vector field $\mathcal{X}_H$). Setting $t = 0$ in this embedding yields an embedding of an affine part of the curve $\mathcal{D}^1_{\mathbf{c}}$. Notice that the components of this embedding are just the residues of the Laurent series $z_0(t), \ldots, z_{35}(t)$ since all these series have a simple pole at worst for $t = 0$. Writing P as a shorthand for (a, b, c, d, e), the resulting map, which we denote by γ_1, is given by

$$\begin{aligned}\gamma_1(P) \quad = \quad & (0 : -1 : 1 : 0_3 : -b : c : 0_2 : b : 0_2 : -c : -e : 0_4 : d : e : \\ & 2ab : 0_2 : -be : 0_3 : cd : 0 : e^2 : b^2e : 0_2 : -c^2d : -2abe)\ .\end{aligned}$$

We have used the convenient notation 0_i to denote that i successive coordinates are zero. Notice that γ_1 is clearly injective, and so is indeed an embedding of the affine curve. Similarly, the embedding γ_i of $\mathcal{D}^i_{\mathbf{c}}$ is found by substituting the corresponding principal balance in the embedding functions $z_0, \ldots, z_{35}$; to determine this principal balance, it suffices to do a cyclic permutation in (3.6) of the indices of the variables x_j, just replacing x_1 by x_i and so on. The five other embeddings that one obtains are given by

$$\begin{aligned}\gamma_2(P) \quad = \quad & (0_2 : -1 : 1 : 0_3 : -b : c : 0_2 : b : 0_2 : d : -e : 0_5 : e : \\ & 2ab : 0_2 : -be : 0_3 : cd : -d^2 : e^2 : b^2e : 0_2 : -ed)\ , \\ \gamma_3(P) \quad = \quad & (0_3 : -1 : 1 : 0 : c : 0 : -b : -c : 0_2 : b : 0_2 : d : -e : 0_3 : -c(b+c) : \\ & 0 : e : 2ab : cd : 0 : -be : 0_3 : -c^2d : -d^2 : e^2 : b^2e : 0 : c^2d)\ , \\ \gamma_4(P) \quad = \quad & (0_4 : -1 : 1 : -b : c : 0_2 : -c : 0_2 : b : 0_2 : d : -e : 0_3 : -c(c+b) : \\ & 0 : e : 0 : cd : 0 : -be : 0_3 : -c^2d : -d^2 : e^2 : b^2e : 0)\ , \\ \gamma_5(P) \quad = \quad & (0_5 : -1 : 0 : -b : c : 0_2 : -c : 0_5 : d : -e : 0_3 : -c(b+c) : \\ & 0_3 : cd : 0 : -be : 0_3 : -c^2d : -d^2 : e^2 : 0)\ , \\ \gamma_6(P) \quad = \quad & (0 : 1 : 0_4 : c : 0 : -b : b : 0_2 : -c : 0_5 : d : -e : 2ab : 0_2 : \\ & -c(c+b) : 0_3 : cd : 0 : -be : b^2e : 0_2 : -c^2d : -d^2 : bce)\ .\end{aligned}$$

It is easy to see that the different images of these embeddings do not intersect. For example, $\mathrm{Im}(\gamma_1)$ and $\mathrm{Im}(\gamma_2)$ cannot intersect because all points in $\mathrm{Im}(\gamma_1)$ have their second coordinate different from zero, while that coordinate vanishes for all points of $\mathrm{Im}(\gamma_2)$.

Since the polynomials $z_0, \ldots, z_{35}$ provide (upon restriction) an embedding of $\mathbf{T_c}$, the embeddings $\gamma_1, \ldots, \gamma_6$ of the affine curve can be holomorphically extended to its smooth compactification; as we will see, the extension is an embedding of the complete curve, so that the six image curves are nonsingular, but these image curves will intersect in several points according to a pattern which we will determine.

To do this, recall from Section 5 that we have determined parametrizations of a neighborhood of each one of points at infinity $\infty_1, \ldots, \infty_5$ of the

	$\mathcal{D}_{\mathbf{c}}^1$	$\mathcal{D}_{\mathbf{c}}^2$	$\mathcal{D}_{\mathbf{c}}^3$	$\mathcal{D}_{\mathbf{c}}^4$	$\mathcal{D}_{\mathbf{c}}^5$	$\mathcal{D}_{\mathbf{c}}^6$
$\mathcal{D}_{\mathbf{c}}^1$	$=$	P_{26}, P_{13}	P_{13}	P_{14}, P'_{14}	P_{15}	P_{26}, P_{15}
$\mathcal{D}_{\mathbf{c}}^2$	P_{26}, P_{13}	$=$	P_{13}, P_{24}	P_{24}	P_{25}, P'_{25}	P_{26}
$\mathcal{D}_{\mathbf{c}}^3$	P_{13}	P_{13}, P_{24}	$=$	P_{24}, P_{35}	P_{35}	P_{36}, P'_{36}
$\mathcal{D}_{\mathbf{c}}^4$	P_{14}, P'_{14}	P_{24}	P_{24}, P_{35}	$=$	P_{35}, P_{46}	P_{46}
$\mathcal{D}_{\mathbf{c}}^5$	P_{15}	P_{25}, P'_{25}	P_{35}	P_{35}, P_{46}	$=$	P_{46}, P_{15}
$\mathcal{D}_{\mathbf{c}}^6$	P_{26}, P_{15}	P_{26}	P_{36}, P'_{36}	P_{46}	P_{46}, P_{15}	$=$

TABLE 3. The irreducible components $\mathcal{D}_{\mathbf{c}}^i$ of the Painlevé divisor intersect in two points which may coincide, in which case the two components are tangent.

Painlevé curves $\Delta_{\mathbf{c}}$. If we substitute the parametrization of one of these points ∞_i in either one of the embeddings γ_j we get an embedding of a punctured neighborhood of ∞_i in $\mathbb{P}^{35}$ and it suffices to let the parameter τ of the parametrization tend to 0 to find the image point in $\mathbb{P}^{35}$. Doing this for the embedding γ_1 we find 5 *different* points, which confirms that the six irreducible components of $\mathcal{D}_{\mathbf{c}}$ are non-singular curves (of genus 2), isomorphic to $\Gamma_{\mathbf{c}}$. Namely, we find the following images:

$$
\begin{aligned}
\infty_1 &\mapsto (0_6 : 1 : 1 : 0_2 : -1 : 0_2 : -1 : 0_7 : \beta - c_1 : 0_9 : -c_3 : 0_2 : -c_4 : 0)\\
\infty_2 &\mapsto (0_6 : 1 : 1 : 0_2 : -1 : 0_2 : -1 : 0_7 : -\beta : 0_9 : -c_3 : 0_2 : -c_4 : 0)\\
\infty_3 &\mapsto (0_{30} : 1 : 0_5)\\
\infty_4 &\mapsto (0_{30} : 1 : -1 : 0_3 : -1)\\
\infty_5 &\mapsto (0_{34} : 1 : 0)
\end{aligned}
$$

When the other embeddings γ_j are used we find in total 30 image points, but they are not all different, as some appear twice and the others three times. In total we find 12 different image points, as indicated in Table 3. The coordinates of the 6 points of the form P_{ij} with $j - i \in \{2, 4\}$ are given by

$$
\begin{aligned}
P_{13} &= (0_{30} : 1 : -1 : 0_3 : -1)\ , & P_{24} &= (0_{31} : 1 : -1 : 0_3)\ ,\\
P_{15} &= (0_{34} : 1 : 0)\ , & P_{26} &= (0_{30} : 1 : 0_5)\ ,\\
P_{35} &= (0_{32} : 1 : -1 : 0_2)\ , & P_{46} &= (0_{33} : 1 : -1 : 0)\ ,
\end{aligned}
$$

while the points $P_{i,i+3}$ have as coordinates

$$
\begin{aligned}
P_{14} &= (0_6 : 1 : 1 : 0_2 : -1 : 0_2 : -1 : 0_7 : -\beta : 0_9 : -c_3 : 0_2 : -c_4 : 0)\,, \\
P_{25} &= (0_7 : 1 : 1 : 0_2 : -1 : 0_{10} : -\beta : 0_9 : -c_4 : 0_3)\,, \\
P_{36} &= (0_6 : 1 : 0 : 1 : -1 : 0_2 : -1 : 0_7 : -\beta : 0_2 : \beta - c_1 : 0_6 : -c_4 : 0_2 : -c_3 : 0 : c_4)\,.
\end{aligned}
$$

For $i = 1, \ldots, 3$, the coordinates of the point $P'_{i,i+3}$ are obtained by replacing β by $c_1 - \beta$ in the coordinates of the point $P_{i,i+3}$ (i.e., replace β, which is a root of the quadratic polynomial $\beta^2 - c_1\beta + c_2$, by the other root). These 12 points are also obtained when substituting the 12 lower balances in the embedding. Namely, the points P_{ij} with $j - i \neq 3$ are obtained by substituting the 6 cyclic permutations of (3.7) in the embedding, the points $P_{i,i+3}$ are obtained similarly by using the 3 cyclic permutations of (3.8) while the points $P'_{i,i+3}$ are obtained as the points $P_{i,i+3}$, but with β replaced by the other root $c_1 - \beta$ of the polynomial $\beta^2 - c_1\beta + c_2$.

It is clear that we have chosen the notations as follows: when $j - i \in \{2, 4\}$ then the point P_{ij} is the unique intersection point of $\mathcal{D}^i_{\mathbf{c}}$ and $\mathcal{D}^j_{\mathbf{c}}$, so the curves $\mathcal{D}^i_{\mathbf{c}}$ and $\mathcal{D}^j_{\mathbf{c}}$ are tangent at P_{ij}; another component passes transversally through this point, namely $\mathcal{D}^{i+1}_{\mathbf{c}}$ in case $j = i + 2$ and $\mathcal{D}^{j+1}_{\mathbf{c}}$ when $j = i + 4$. Also, the points $P_{i,i+3}$ and $P'_{i,i+3}$ are the unique intersection points of $\mathcal{D}^i_{\mathbf{c}}$ and $\mathcal{D}^{i+3}_{\mathbf{c}}$ and no other curve of the divisor passes through them. With this notation, $\mathcal{D}^i_{\mathbf{c}}$ contains the points $P_{i,i+2}$, $P_{i-2,i}$, $P_{i-1,i+1}$, $P_{i,i+3}$ and $P'_{i,i+3}$.

Though the table contains all information on the intersection pattern of the curves $\mathcal{D}^i_{\mathbf{c}}$ a few pictures may help to visualize this pattern. First, here is the intersection pattern of $\mathcal{D}^1_{\mathbf{c}}$ with a neighbor, a second nearest neighbor and its farest neighbor.

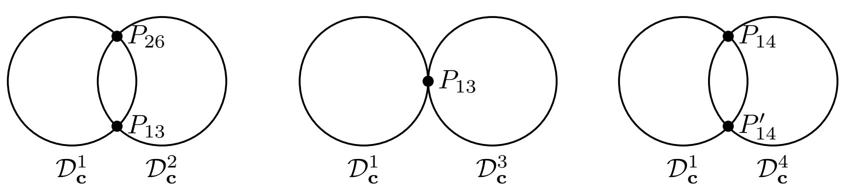

FIGURE 1. Each Painlevé curve $\mathcal{D}^i$ is tangent to its second nearest neighbors and intersects the other Painlevé curves in two points.

Secondly, we display the intersection pattern of $\mathcal{D}^1_{\mathbf{c}}$ with two other curves. There are three essentially different possibilities, according to whether the configuration contains three, two or no consecutive curves.

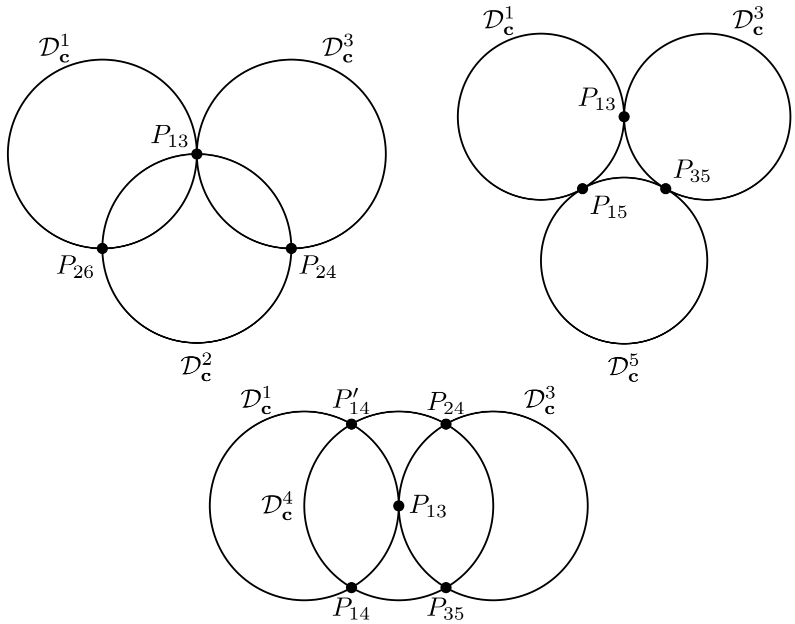

FIGURE 2. For three Painlevé curves there are three possible intersection patterns.

It is also instructive to picture one single $\mathcal{D}_{\mathbf{c}}$, say $\mathcal{D}^1_{\mathbf{c}}$, with its 5 points at infinity, as well as arcs of the other $\mathcal{D}^i_{\mathbf{c}}$ passing through them:

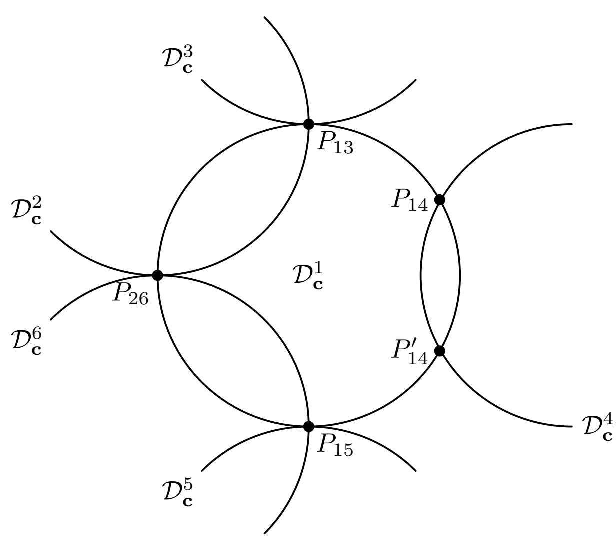

FIGURE 3. The intersection pattern of one of the Painlevé curves with all the other Painlevé curves.

We compare this configuration to a similar configuration of the divisor needed for another natural — but singular — compactification of the generic fiber $\mathbf{F_c}$ of the momentum map $\mathbf{F}:\mathbb{C}^6\to\mathbb{C}^4$. It is obtained by introducing an extra variable x_0 and making the affine equations for $\mathbf{F_c}$ homogeneous, i.e., to consider the compact surface $\bar{\mathbf{F}}_\mathbf{c}\subset\mathbb{P}^6$, given by

$$\begin{aligned}
&x_1+x_2+x_3+x_4+x_5+x_6=c_1x_0\ ,\\
&x_1x_4+x_2x_5+x_3x_6+x_1x_3+x_2x_4+x_3x_5+x_4x_6+x_1x_5+x_2x_6=c_2x_0^2\ ,\\
&x_1x_3x_5=c_3x_0^3\ ,\\
&x_2x_4x_6=c_4x_0^3\ .
\end{aligned}\tag{6.1}$$

Since $\mathbf{c}$ is generic in the sense that the affine part $\mathbf{F_c}$ of $\bar{\mathbf{F}}_\mathbf{c}$ is non-singular, all singularities of $\mathbf{F_c}$ are at infinity, i.e., are contained in the divisor $\mathcal{C}:=\bar{\mathbf{F}}_\mathbf{c}\setminus\mathbf{F_c}$. Equations for $\mathcal{C}$ are obtained by intersecting $\bar{\mathbf{F}}_\mathbf{c}$ with the hyperplane $x_0=0$, giving the following equations (they are independent of $\mathbf{c}$, which is the reason why we do not add an index $\mathbf{c}$ to $\mathcal{C}$):

$$\begin{aligned}
&x_1+x_2+x_3+x_4+x_5+x_6=0\ ,\\
&x_1x_4+x_2x_5+x_3x_6+x_1x_3+x_2x_4+x_3x_5+x_4x_6+x_1x_5+x_2x_6=0\ ,\\
&x_1x_3x_5=0\ ,\\
&x_2x_4x_6=0\ .
\end{aligned}\tag{6.2}$$

Starting from the last two equations, it is clear how to determine the irreducible components of the divisor $\mathcal{C}$: we need to pick an odd index i and an even index j and set $x_i=x_j=0$ in the other two equations which are easily rewritten as a single quadratic equation in five variables, so they are conics. Since each choice of i and j leads to a different conic, we get 9 conics in total. For example, setting $x_1=x_2=0$ we get the following non-singular conic:

$$\mathcal{C}^1:\left\{\begin{array}{l} x_3+x_4+x_5+x_6=0\ ,\\ x_3x_6+x_3x_5+x_4x_6=0\ .\end{array}\right.\tag{6.3}$$

In view of the order 6 automorphism there are six such conics, which we denote by $\mathcal{C}^i$, with $i=1,\dots,6$, where $\mathcal{C}^i$ is the conic contained in the subspace $x_i=x_{i+1}=0$. The three remaining conics are obtained by setting $x_i=x_{i+3}=0$. For example, setting $x_3=x_6=0$ we get the following conic:

$$\mathcal{L}^1:\left\{\begin{array}{l} x_1+x_2+x_4+x_5=0\ ,\\ (x_1+x_2)(x_4+x_5)=0\ .\end{array}\right.$$

The conic is degenerate, consisting of the double line $x_1+x_2=x_4+x_5=0$. Two other such lines are obtained by using the order 6 automorphism. They are denoted by $\mathcal{L}^i$ where $\mathcal{L}^i$ is the (double) line contained in the subspace $x_{i+2}=x_{i-1}=0$.

The singularities of $\mathbf{F_c}$ which are contained in $\mathcal{C}$ are the points $(0:x_1:x_2:\cdots:x_6)$ where the rank of the following Jacobian matrix is at most 3:

$$\begin{pmatrix} -c_1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & x_3+x_4+x_5 & x_4+x_5+x_6 & x_1+x_5+x_6 & x_1+x_2+x_6 & x_1+x_2+x_3 & x_2+x_3+x_4 \\ 0 & x_3x_5 & 0 & x_1x_5 & 0 & x_1x_3 & 0 \\ 0 & 0 & x_4x_6 & 0 & x_2x_6 & 0 & x_2x_4 \end{pmatrix}.$$

Consider the following parametrization of $\mathcal{L}^1$:

$$(u:v) \mapsto (0:u:-u:0:v:-v:0) \tag{6.4}$$

and substitute it in the Jacobian matrix, to see that except for two columns, all columns are a multiple of the first column, and so the rank is at most 3 (it is in fact 3 at all points where $uv \neq 0$; the rank drops to 2 at the points where $u = 0$ or $v = 0$). By symmetry, the same holds true for the lines $\mathcal{L}^2$ and $\mathcal{L}^3$. The suface $\mathbf{F_c}$ is therefore singular at all points of the three lines $\mathcal{L}^1, \dots, \mathcal{L}^3$. With some extra work it can be shown that $\mathbf{F_c}$ has for generic $\mathbf{c}$ no other singularities.

Because of the simple equations for the conics and lines it is easy to determine how they intersect. It is clear that the lines $\mathcal{L}^i$ do not intersect. In order to find out how $\mathcal{C}^1$ and $\mathcal{L}^1$ intersect, we set $x_1 = x_2 = 0$ in the parametrization (6.4) of $\mathcal{L}^1$ to find that $u = 0$, yielding $Q_1 := (0_4 : 1 : -1 : 0)$ as the unique intersection point. Since the tangent line to the conic $\mathcal{C}^1$ at this intersection point has the parametrization $(u:v) \mapsto (0_3 : u : -v : v-2u : u)$ it is different from $\mathcal{L}^1$, so that the point $(0_4 : 1 : -1 : 0)$ is a simple intersection point; the latter fact also follows from the fact that the line $\mathcal{L}^1$ is not contained in the subspace $x_1 = x_2 = 0$ containing the conic $\mathcal{C}^1$.

The intersection points between the conics and lines are given in Table 4. There are 6 intersection points which we denote by $Q_1, \dots, Q_6$, where Q_i is the unique intersection point of $\mathcal{C}^i$ and $\mathcal{L}^i$ (or $\mathcal{L}^{i-3}$ when $i > 3$). The coordinates of Q_i are all zero, except for the $(i+3)$-th and $(i+4)$-th which are equal with opposite sign. Notice that each line $\mathcal{L}^i$ contains two of these special points, to wit Q_i and Q_{i+3} and that each conic contains three of them, to wit Q_{i-1}, Q_i and Q_{i+1}.

	$\mathcal{C}^1$	$\mathcal{C}^2$	$\mathcal{C}^3$	$\mathcal{C}^4$	$\mathcal{C}^5$	$\mathcal{C}^6$
$\mathcal{L}^1$	Q_1	Q_1	Q_4	Q_4	Q_4	Q_1
$\mathcal{L}^2$	Q_2	Q_2	Q_2	Q_5	Q_5	Q_5
$\mathcal{L}^3$	Q_6	Q_3	Q_3	Q_3	Q_6	Q_6

TABLE 4. Each conic $\mathcal{C}^i$ intersects each line $\mathcal{L}^j$ transversally in a single point.

One determines in a similar way how the conics intersect. For example, to determine how $\mathcal{C}^1$ and $\mathcal{C}^2$ intersect one sets $x_3 = 0$ in (6.3) to find two intersection points, namely $Q_1 = (0:0:0:0:1:-1:0)$ and $Q_2 = (0:0:0:0:0:1:-1)$. The intersection points of the curves $\mathcal{C}^i$ are indicated in Table 5. Notice that the curves $\mathcal{C}^i$ and $\mathcal{C}^{i+3}$ are disjoint, and that the conics intersect (only) at the intersection points $Q_1, \dots, Q_6$ of the conics and the lines. Since every conic $\mathcal{C}^i$ contains three of the points Q_j and through every point Q_j pass three of the conics $\mathcal{C}^i$, these 6 conics and 6 points form a 6_3 configuration.

	$\mathcal{C}^1$	$\mathcal{C}^2$	$\mathcal{C}^3$	$\mathcal{C}^4$	$\mathcal{C}^5$	$\mathcal{C}^6$
$\mathcal{C}^1$	$=$	Q_1, Q_2	Q_2	—	Q_6	Q_1, Q_6
$\mathcal{C}^2$	Q_1, Q_2	$=$	Q_2, Q_3	Q_3	—	Q_1
$\mathcal{C}^3$	Q_2	Q_2, Q_3	$=$	Q_3, Q_4	Q_4	—
$\mathcal{C}^4$	—	Q_3	Q_3, Q_4	$=$	Q_4, Q_5	Q_5
$\mathcal{C}^5$	Q_6	—	Q_4	Q_4, Q_5	$=$	Q_5, Q_6
$\mathcal{C}^6$	Q_1, Q_6	Q_1	—	Q_5	Q_5, Q_6	$=$

TABLE 5. The non-singular conics $\mathcal{C}^i$ are either disjoint, they intersect transversally in one point or they intersect in two points.

In order to better visualize the configuration of conics and lines, we represent them in the following picture, where we draw for the line $\mathcal{L}^1$ and for the conic $\mathcal{L}^1$ all lines and conics passing through them. This picture is to be compared with Figure 3, which represents the Painlevé divisor, which appears in the compactification of $\mathbf{F_c}$ into an Abelian variety. It would be interesting to obtain the latter compactification — including a complete description of the divisor added in the process of compactification — by purely algebraic geometric means, i.e., without using the periodic 6-particle KM vector field which we have used in a very essential way.

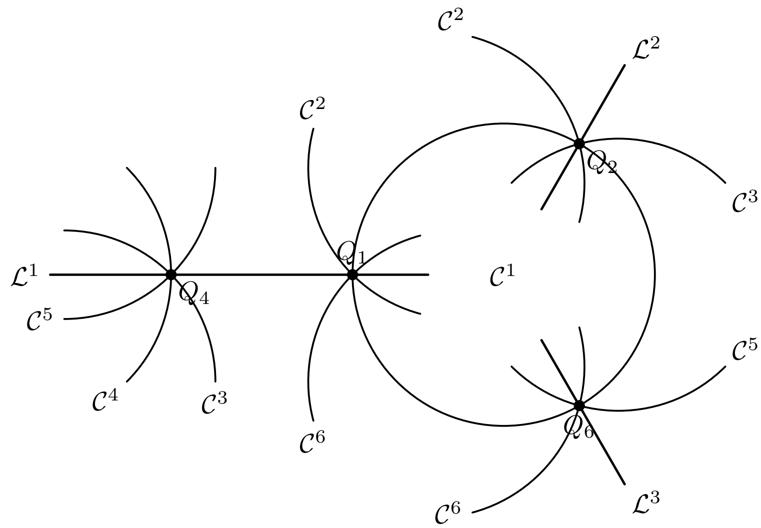

FIGURE 4. The intersection pattern of one of the lines and one of the conics with all other lines and conics.

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