

LOTKA-VOLTERRA SYSTEMS SATISFYING A STRONG PAINLEVÉ PROPERTY

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ABSTRACT. We use a strong version of the Painlevé property to discover and characterize a new class of n -dimensional Hamiltonian Lotka-Volterra systems, which turn out to be Liouville integrable as well as superintegrable. These systems are in fact Nambu systems, they possess Lax equations and they can be explicitly integrated in terms of elementary functions. We apply our analysis to systems containing only quadratic nonlinearities of the form $a_{ij}x_ix_j, i \neq j$, and require that all variables diverge as t^{-1} . We also require that the leading terms depend on $n - 2$ free parameters. We thus discover a cocycle relation among the coefficients a_{ij} of the equations of motion and by integrating the cocycle equations we show that they are equivalent to the above strong version of the Painlevé property. We also show that these systems remain explicitly solvable even if a linear term b_ix_i is added to the i -th equation, even though this violates the Painlevé property, as logarithmic singularities are introduced in the Laurent solutions, at the first terms following the leading order pole.

1. INTRODUCTION

Let m and n be arbitrary integers, with $1 < m \leq n$. We consider on \mathbb{C}^n the Lotka-Volterra system

$$\dot{x}_i = x_i \sum_{\lambda=1}^m a_{i\lambda} x_\lambda, \quad (i = 1, \dots, n), \quad (1.1)$$

where $A = (a_{i\lambda})$ is an $n \times m$ matrix with complex entries, which is skew-symmetric in the sense that $a_{\lambda\mu} = -a_{\mu\lambda}$ for $1 \leq \lambda, \mu \leq m$. The system (1.1) is a Hamiltonian system, whose Hamiltonian is the linear function $H = x_1 + x_2 + \dots + x_m$ with respect to a family of compatible quadratic Poisson structures, defined by the following brackets:

$$\{x_i, x_j\} := a_{ij}x_ix_j, \quad 1 \leq i < j \leq n, \quad (1.2)$$

where the constants a_{ij} with $m < i < j \leq n$ can be picked arbitrarily (the other constants a_{ij} are then determined by the skew-symmetry relation $a_{ij} = -a_{ji}$). As is well-known, (1.2) defines a Poisson bracket (i.e., the Jacobi identity is automatically satisfied). For more information on these Poisson structures, which are often called *diagonal* or *log-canonical*, see [11, Section 8.2]. Of course many other choices for the coefficients in eq. (1.1) are available and some are known to be integrable. This is the case, for example, with the so-called Projective Riccati Equations, which have been integrated through the use of superposition principles [4].

2010 *Mathematics Subject Classification.* 37J35, 39A22.

Key words and phrases. Integrable Lotka Volterra systems, strong Painlevé property.

In this paper, we wish to select those matrices A for which (1.1) satisfies a “strong” Painlevé property, that will be described below. First, we recall that the Painlevé (or P-) property for a system such as (1.1) amounts to the requirement that all its solutions be single-valued and meromorphic in the sense that all movable singularities are poles, about which the solutions can be expanded as (convergent) Laurent series, depending on $n - 1$ free parameters (the n -th of them being the location of the singularity t_*). These Laurent solutions are called *principal balances*, while the Laurent solutions which depend on fewer free parameters are called *lower balances*.

It was S. Kovalevskaya who first used this criterion to select from the class of all tops the ones that ought to be integrable, leading to the discovery of a new integrable case, which now bears her name (see [1, Section 10.1.2]). A little later, in the early 1900’s, P. Painlevé developed this approach into a systematic theory that enabled him to identify all 50 second order ordinary differential equations (ODEs) that possess what we call the P-property [3, 14]. 44 of them were found to be integrable and solvable in terms of elementary functions, while 6 were shown not to be reducible to first order ODEs and were solved by the so-called Painlevé transcendental functions [10].

For a proof that the existence of principal balances is a necessary condition for algebraic integrability, see [1, Section 6.2]. In this paper, we impose on (1.1) the following two conditions, which constitute what we call the “strong” P-property: (a) The system has principal balances where each variable is expressed as a Laurent series that starts with a simple pole, and (b) $n - 2$ free parameters appear at the leading order term of these balances.

We show in Proposition 2.2 that if (1.1) satisfies (a) and (b) above then the constants $a_{i\lambda}$ can be written as $a_{i\lambda} = a_\lambda - a_i$ for some constants a_1, \dots, a_n , implying that (1.1) possesses several properties such as Liouville integrability and superintegrability (Proposition 3.1), the fact that it is a Nambu system (Proposition 3.2), that it can be integrated in terms of elementary functions (Proposition 3.3) and that it is given by Lax equations (Proposition 3.4). Conversely, when the constants $a_{i\lambda}$ are of the form $a_{i\lambda} = a_\lambda - a_i$ for some constants a_1, \dots, a_n , then (1.1) satisfies (a) and (b) above, hence the strong Painlevé property that we impose on (1.1) is actually equivalent to a natural collection of cocycle conditions on the coefficients $a_{i\lambda}$, leading to a new integrable family of Lotka-Volterra systems, having many nice features.

2. LOTKA-VOLTERRA SYSTEMS AND THE PAINLEVÉ PROPERTY

Throughout this paper, m and n are arbitrary integers, with $1 < m \leq n$. We consider on \mathbb{C}^n the Lotka-Volterra system

$$\dot{x}_i = x_i \sum_{\lambda=1}^m a_{i\lambda} x_\lambda, \quad (i = 1, \dots, n), \quad (2.1)$$

where $A = (a_{i\lambda})$ is an $n \times m$ matrix with complex entries, which is skew-symmetric in the sense that $a_{\lambda\mu} = -a_{\mu\lambda}$ for $1 \leq \lambda, \mu \leq m$. In these formulas, and in what follows, we use latin letters i, j, \dots for indices which belong to the range $1, \dots, n$, while we use greek letters λ, μ, \dots for indices belonging to the range $1, \dots, m$. We recall from the introduction the following two conditions, which we will impose on the system (2.1):

- (P1) The system has principal balances where each variable is expressed as a Laurent series that starts with a simple pole.
- (P2) $n - 2$ free parameters appear at the leading order term of these balances.

We analyze these conditions and translate them into conditions on the entries of the matrix A . Condition (P1) means that there exists a collection of n complex Laurent series of the form

$$x_i(t) = \frac{1}{\tau} (x_i^{(0)} + x_i^{(1)}\tau + x_i^{(2)}\tau^2 + \dots), \quad \tau = t - t_*, \quad (i = 1, \dots, n), \quad (2.2)$$

which is a solution to (2.1) and where all leading coefficients $x_i^{(0)}$ are different from zero. According to [1, Theorem 7.25] such a solution is always convergent (for small non-zero τ). A direct substitution of (2.2) in (2.1) shows that the leading coefficients $x_i^{(0)}$ satisfy the quadratic equations

$$-x_i^{(0)} = x_i^{(0)} \sum_{\lambda=1}^m a_{i\lambda} x_\lambda^{(0)}, \quad (i = 1, \dots, n). \quad (2.3)$$

Since all $x_i^{(0)}$ are non-zero, the latter equations are equivalent to the following linear system:

$$-1 = \sum_{\lambda=1}^m a_{i\lambda} x_\lambda^{(0)}, \quad (i = 1, \dots, n). \quad (2.4)$$

Notice that the $n - m$ variables $x_j^{(0)}$ with $m < j \leq n$ are absent from the latter equations. It follows that, if all free parameters in the principal balances, except one, appear at this step, then the solution space of (2.4) is $(m - 2)$ -dimensional, so that A has rank 2. In the following proposition we give an explicit description of all such $n \times m$ matrices A , having the additional property that (2.4) has an $(m - 2)$ -dimensional solution space.

Proposition 2.1. *Let $A = (a_{i\lambda})$ be an $n \times m$ matrix, where $1 < m \leq n$, and let $b \in \mathbb{C}^*$. We assume that the upper square part of A is skew-symmetric, i.e.,*

$a_{\lambda\mu} = -a_{\mu\lambda}$ for $1 \leq \lambda, \mu \leq m$. Denote by B the column vector of size n whose entries are all equal to b . The following conditions are equivalent:

- (i) A has rank 2 and the equation $AX = B$ has a solution;
- (ii) The upper square part of A is non-zero and for every λ, μ, i with $1 \leq \lambda < \mu \leq m$ and $1 \leq i \leq n$ the cocycle condition $a_{i\mu} = a_{i\lambda} + a_{\lambda\mu}$ holds;
- (iii) There exist constants a_1, \dots, a_n , with a_1, \dots, a_m not all equal, such that $a_{i\lambda} = a_\lambda - a_i$ for all λ, i with $1 \leq i \leq n$ and $1 \leq \lambda \leq m$.

Proof. By homogeneity, we may assume that $b = 1$. We first show that (i) implies (ii). Suppose that A has rank 2 and its upper square part is skew-symmetric. Let us denote by $[A, B]$ the concatenation of the matrix A and the column vector B . The equation $AX = B$ has a solution if and only if the rank of $[A, B]$ is the same as the rank of A , which is equal to 2. Let $1 \leq \lambda < \mu < \nu \leq m$ and consider the following submatrix of $[A, B]$:

$$A_{\lambda, \mu, \nu} := \begin{pmatrix} 0 & a_{\lambda\mu} & a_{\lambda\nu} & 1 \\ -a_{\lambda\mu} & 0 & a_{\mu\nu} & 1 \\ -a_{\lambda\nu} & -a_{\mu\nu} & 0 & 1 \end{pmatrix}.$$

The rank of the matrix $A_{\lambda, \mu, \nu}$ is at most 2 if and only if all its 3×3 minors vanish, which is equivalent to the equations

$$\begin{aligned} a_{\lambda\mu}(a_{\lambda\mu} + a_{\mu\nu} - a_{\lambda\nu}) &= 0, \\ a_{\mu\nu}(a_{\lambda\mu} + a_{\mu\nu} - a_{\lambda\nu}) &= 0, \\ a_{\lambda\nu}(a_{\lambda\mu} + a_{\mu\nu} - a_{\lambda\nu}) &= 0. \end{aligned}$$

In turn, this is equivalent to the single condition

$$a_{\lambda\mu} + a_{\mu\nu} + a_{\nu\lambda} = 0, \tag{2.5}$$

(recall that the upper square part of A is skew-symmetric). This shows that (i) implies $a_{\nu\mu} = a_{\nu\lambda} + a_{\lambda\mu}$, for $1 \leq \lambda, \mu, \nu \leq m$ (i.e., for the entries of the upper square part of A). If $m = n$ this shows that (i) implies the second part of (ii). Suppose therefore that $m < n$ and let λ, μ, i be such that $1 \leq \lambda < \mu \leq m < i \leq n$. If $a_{\lambda\mu} \neq 0$ then, as above (considering the matrix $A_{\lambda, \mu, i}$) we get that $a_{\lambda\mu} - a_{i\mu} + a_{i\lambda} = 0$, as wanted. If $a_{\lambda\mu} = 0$, then there exists a ν with $1 \leq \nu \leq m$ such that $a_{\lambda\nu} \neq 0$ (and hence, $a_{\mu\nu} \neq 0$, thanks to the cocycle relation (2.5). Indeed, if $a_{\lambda\mu} = 0$ then $a_{\lambda\nu} = a_{\mu\nu}$): indeed, for a fixed λ not all $a_{\lambda\nu}$ can be zero, because otherwise $AX = B$ would not have a solution. As above, the fact that $a_{\lambda\nu} \neq 0$ and $a_{\mu\nu} \neq 0$ implies that

$$\begin{aligned} a_{\lambda\nu} - a_{i\nu} + a_{i\lambda} &= 0, \\ a_{\mu\nu} - a_{i\nu} + a_{i\mu} &= 0. \end{aligned}$$

Subtracting these two equations and using the cocycle condition (2.5), we find that $a_{i\lambda} = a_{i\mu} + a_{\mu\lambda}$. This shows that (i) implies the second part of (ii); the first part

of (ii) is an immediate consequence of (i) because if $AX = B$ has a solution then every line of A is non-zero. Suppose now that the entries of A satisfy the cocycle conditions $a_{i\mu} = a_{i\lambda} + a_{\lambda\mu}$, where $1 \leq \lambda < \mu \leq m$ and $1 \leq i \leq n$. Choose a_1 arbitrarily and define a_i for $1 < i \leq n$ by $a_i := a_1 - a_{i1}$. Then we have, for $1 \leq i \leq n$ and $1 \leq \lambda \leq m$,

$$a_{i\lambda} = a_{i1} + a_{1\lambda} = a_1 - a_i + a_\lambda - a_1 = a_\lambda - a_i ,$$

which shows the existence of the constants a_1, \dots, a_n , announced in (iii). Notice that these constants are uniquely determined, once one of the constants (a_1 , for example) has been fixed. Since the upper square part of A is non-zero, there exist λ, μ with $1 \leq \lambda, \mu \leq m$, such that $a_{\lambda\mu} = a_\mu - a_\lambda \neq 0$, which implies that a_1, \dots, a_m are not all equal. This shows that (ii) implies (iii). It remains to be shown that (iii) implies (i): suppose that the entries of the matrix A are of the form $a_{i\lambda} = a_\lambda - a_i$ ($1 \leq i \leq n$ and $1 \leq \lambda \leq m$) and that a_1, \dots, a_m are not all equal (so that the upper square part of A is non-zero). We first show that the rank of A is equal to 2. To do this, it suffices to show that all 3×3 minors of A vanish. Any 3×3 submatrix of A is of the form

$$A_{i,j,k}^{\lambda,\mu,\nu} := \begin{pmatrix} a_\lambda - a_i & a_\mu - a_i & a_\nu - a_i \\ a_\lambda - a_j & a_\mu - a_j & a_\nu - a_j \\ a_\lambda - a_k & a_\mu - a_k & a_\nu - a_k \end{pmatrix}$$

for some $1 \leq i < j < k \leq n$ and $1 \leq \lambda < \mu < \nu \leq m$. It can be checked by direct computation that the determinant of $A_{i,j,k}^{\lambda,\mu,\nu}$ is zero; for a quicker proof, observe that when one subtracts the first row of $A_{i,j,k}^{\lambda,\mu,\nu}$ from its second and third rows, the new second and third rows are proportional. This shows that all 3×3 minors of A vanish, so that A is of rank two (recall that the upper square part of A is non-zero and skew-symmetric). For an alternative proof that A is of rank two, observe that A is the difference of two $n \times m$ matrices of rank one:

$$A = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} (a_1 \ a_2 \ \dots \ a_m) - \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} (1 \ 1 \ \dots \ 1) .$$

This shows the first part of (i). In order to show that the equation $AX = B$ has a solution (still taking $b = 1$), pick λ, μ such that $1 \leq \lambda < \mu \leq m$ and such that $a_{\lambda\mu} \neq 0$. Then a particular solution of $AX = B$ is given by

$$x_\lambda = -x_\mu = \frac{1}{a_{\lambda\mu}} , \quad x_\nu = 0 \text{ for } \nu \in \{1, \dots, m\} \setminus \{\lambda, \mu\} . \quad (2.6)$$

For future use, notice that for any $\nu \in \{1, \dots, m\} \setminus \{\lambda, \mu\}$ a solution to the homogeneous equation $AX = 0$ is given by

$$x_\lambda = a_{\mu\nu} , \quad x_\mu = a_{\nu\lambda} , \quad x_\nu = a_{\lambda\mu} , \quad (2.7)$$

$$x_\rho = 0 \text{ for } \rho \in \{1, \dots, m\} \setminus \{\lambda, \mu, \nu\} . \quad (2.8)$$

Indeed, with this choice of X , the i -th entry of the vector AX is given by

$$a_{i\lambda}a_{\mu\nu} + a_{i\mu}a_{\nu\lambda} + a_{i\nu}a_{\lambda\mu} = (a_{i\lambda} - a_{i\nu})a_{\mu\nu} + (a_{i\mu} - a_{i\nu})a_{\nu\lambda} = 0.$$

This shows that (iii) implies (i). \square

According to Proposition 2.1, the only systems (2.1) which satisfy (P1) and (P2) are defined by $n \times m$ matrices $A = (a_{i\lambda})$ satisfying the cocycle conditions $a_{i\mu} = a_{i\lambda} + a_{\lambda\mu}$ for $1 \leq \lambda < \mu \leq m$ and $1 \leq i \leq n$, and whose upper square part is non-zero. Let us show that for any such matrix, (P1) and (P2) are satisfied. We have already shown at the end of the proof of the proposition (see (2.6) — (2.8)) how to construct the complete solution to the equation $AX = B$, which gives the following solution to the indicial equations (2.3):

$$\begin{aligned} x_\lambda^{(0)} &= \frac{1}{a_{\lambda\mu}} + \sum_{\nu \neq \lambda, \mu} \alpha_\nu a_{\mu\nu}, \\ x_\mu^{(0)} &= \frac{1}{a_{\mu\lambda}} + \sum_{\nu \neq \lambda, \mu} \alpha_\nu a_{\nu\lambda}, \\ x_i^{(0)} &= \alpha_i a_{\lambda\mu}, \quad (i \in \{1, \dots, n\} \setminus \{\lambda, \mu\}), \end{aligned} \tag{2.9}$$

where α_i is a free parameter, for $i \in \{1, \dots, n\} \setminus \{\lambda, \mu\}$. The existence and uniqueness of the subsequent terms in the series (2.2) is governed by the Kowalevski matrix K , whose entries K_{ij} are given by

$$K_{ij} = \left(\frac{\partial f_i}{\partial x_j}(x^{(0)}) + \delta_{i,j} \right), \quad (1 \leq i, j \leq n),$$

where f_i stands for the right hand side of (2.1), to wit

$$f_i = x_i \sum_{\lambda=1}^m a_{i\lambda} x_\lambda,$$

and $\delta_{i,j}$ is the Kronecker delta. Explicitly, the entries of K are given by

$$K_{i,i} = \sum_{\lambda=1}^m a_{i\lambda} x_\lambda^{(0)} + 1 = 0, \quad (i = 1, \dots, n), \tag{2.10}$$

$$K_{i,\lambda} = a_{i\lambda} x_i^{(0)}, \quad (i = 1, \dots, n, \lambda = 1, \dots, m, \lambda \neq i), \tag{2.11}$$

$$K_{i,j} = 0, \quad (i = 1, \dots, n, j = m+1, \dots, n, i \neq j), \tag{2.12}$$

where for the computation of $K_{i,i}$ we have used (2.4). We claim that the characteristic polynomial of K is given by $\chi(K, \lambda) = \lambda^{n-2}(\lambda^2 - 1)$. To show this, first notice that (2.10) — (2.12) can be combined in the single formula $K_{i,j} = a_{ij} x_i^{(0)}$, valid for all i, j , if we define $a_{ij} := 0$ for $1 \leq i \leq n$ and $m < j \leq n$. Since A has rank two, it implies that K has at most rank two, and hence $\chi(K, \lambda)$ is divisible by λ^{n-2} . Since (2.1) is homogeneous, -1 is a root of $\chi(K, \lambda)$ (see [1, Proposition 7.11]). Finally, (2.10) trivially implies that K has trace zero, so the sum of all roots of $\chi(K, \lambda)$ is zero, showing that the last root of $\chi(K, \lambda)$ is 1. This validates our claim.

In order to complete the proof that our systems satisfy (P1) and (P2), it remains to be shown that the degenerate linear system which is obtained when determining $x_i^{(1)}$ has a solution; then this solution will depend on a free parameter, so the first two terms of the Laurent solution depend on $n - 1$ free parameters, and all other terms are uniquely determined by these first two terms, leading to a principal balance. By substituting the first two terms of the Laurent series (2.2) into (2.1) we find that the equation that the variables $x_i^{(1)}$ have to satisfy are actually homogeneous: for $i = 1, \dots, n$ they need to satisfy the linear equations

$$\sum_{\lambda=1}^m a_{i\lambda} x_i^{(0)} x_\lambda^{(1)} + x_i^{(1)} x_\lambda^{(0)} = 0. \quad (2.13)$$

Obviously, this system has a solution, hence a free parameter appears at this step and we are done. An alternative way to see this is to observe that the matrix of the homogeneous system yielding the $x_i^{(1)}$ has determinant zero, since adding all its rows yields a row of zeros due to the relations satisfied by the $x_i^{(0)}$ coefficients.

We summarize what we have proved in the following proposition:

Proposition 2.2. *Let $A = (a_{i\lambda})$ be a $n \times m$ matrix with complex entries, where $1 < m \leq n$. It is assumed that A is skew-symmetric in the sense that $a_{\lambda\mu} = -a_{\mu\lambda}$ for $1 \leq \lambda, \mu \leq m$. . The Lotka-Volterra system*

$$\dot{x}_i = x_i \sum_{\lambda=1}^m a_{i\lambda} x_\lambda, \quad (i = 1, \dots, n), \quad (2.14)$$

satisfies the properties (P1) and (P2) if and only if the entries of A satisfy the cocycle conditions $a_{i\mu} = a_{i\lambda} + a_{\lambda\mu}$, for $1 \leq \lambda, \mu \leq m$ and $1 \leq i \leq n$; in turn, these conditions are equivalent to the existence of constants a_1, \dots, a_n , with a_1, \dots, a_m not all equal, and such that $a_{i\lambda} = a_\lambda - a_i$ for all such i, λ .

For $n = 2$ (so that $m = 2$) the cocycle conditions are automatically satisfied and so (P1) and (P2) always hold. For $n = 3$ (so that $m = 2$ or $m = 3$), the rank of A is automatically equal to two, so that (P1) already implies the cocycle condition $a_{12} + a_{23} = a_{13}$; hence, in this case, (P2) is a consequence of (P1).

In what follows, we will call a Lotka-Volterra system, satisfying the conditions of Proposition 2.2, a *Lotka-Volterra-Painlevé system*.

3. INTEGRABILITY AND EXPLICIT SOLUTIONS

We show in this section that every Lotka-Volterra-Painlevé system, as defined in the previous section, is Liouville integrable and superintegrable. We show that these systems are Nambu systems, have Lax equations and can be explicitly integrated in terms of elementary functions.

Proposition 3.1. *Suppose that*

$$\dot{x}_i = x_i \sum_{\lambda=1}^m a_{i\lambda} x_\lambda, \quad (i = 1, \dots, n), \quad (3.1)$$

is a Lotka-Volterra-Painlevé system. Then (3.1) is Hamiltonian with respect to a Poisson structure $\{\cdot, \cdot\}$ of rank 2, with $H = x_1 + \dots + x_m$ as Hamiltonian. Moreover, this Poisson structure has $n-2$ functionally independent Casimir functions, defined on an open dense subset of \mathbb{C}^n , so that (3.1) is both Liouville and superintegrable.

Proof. In view of Proposition 2.1 there exist constants a_1, \dots, a_n such that $a_{i\lambda} = a_\lambda - a_i$ for $1 \leq i \leq n$ and $1 \leq \lambda \leq m$. We extend A to a skew-symmetric $n \times n$ matrix Π by setting $\pi_{ij} := a_j - a_i$ for $1 \leq i, j \leq n$. According to Proposition 2.1, Π has rank 2. The entries of Π satisfy the cocycle relations

$$\pi_{ij} + \pi_{jk} + \pi_{ki} = 0, \quad (1 \leq i, j, k \leq n), \quad (3.2)$$

which extend the cocycle relations (2.5) satisfied by the entries of A . The Poisson structure, defined by $\{x_i, x_j\} := \pi_{ij} x_i x_j$ for all $1 \leq i, j \leq n$ is a diagonal Poisson structure, hence its rank is the same rank as the rank of Π (see [11, Example 8.14]), i.e., the Poisson structure has rank 2. The vector field (3.1) is Hamiltonian with respect to this Poisson structure, with $H = x_1 + \dots + x_m$ as a Hamiltonian, hence it suffices to exhibit $n-2$ functionally independent Casimirs to show that (3.1) is both Liouville and superintegrable. We construct these from the null vectors of Π ; indeed, if (s_1, s_2, \dots, s_n) is a null vector of Π then the product $C = x^{s_1} x^{s_2} \dots x^{s_n}$ is a Casimir of the Poisson structure defined by Π , because $\{x_i, C\} = \{x_i, x_1^{s_1} x_2^{s_2} \dots x_n^{s_n}\} = \left(\sum_{j=1}^n \pi_{ij} s_j \right) C = 0$ for $i = 1, \dots, n$. In order to construct a basis for the null vectors of Π , choose λ, μ with $1 \leq \lambda, \mu \leq m$ such that $\pi_{\lambda\mu} \neq 0$ and consider for $k \in \{1, \dots, n\} \setminus \{\lambda, \mu\}$ the vector X whose components are defined by

$$x_\lambda = \pi_{\mu k}, \quad x_\mu = \pi_{k\lambda}, \quad x_k = \pi_{\lambda\mu}, \quad (3.3)$$

$$x_\ell = 0 \quad \text{for } \ell \in \{1, \dots, n\} \setminus \{\lambda, \mu, k\}. \quad (3.4)$$

Indeed, with this choice of X , the vanishing of the ℓ -th entry of the vector ΠX can be computed using the cocycle relations (3.2) as follows:

$$\pi_{\ell\lambda} \pi_{\mu k} + \pi_{\ell\mu} \pi_{k\lambda} + \pi_{\ell k} \pi_{\lambda\mu} = (\pi_{\ell\lambda} - \pi_{\ell k}) \pi_{\mu k} + (\pi_{\ell\mu} - \pi_{\ell k}) \pi_{k\lambda} = 0. \quad (3.5)$$

By the above procedure, it leads to the Casimirs $x_\lambda^{a_{\mu k}} x_\mu^{a_{k\lambda}} x_k^{a_{\lambda\mu}}$, where $k \in \{1, \dots, n\} \setminus \{\lambda, \mu\}$; these $n-2$ Casimirs are indeed independent because the latter Casimir is the only one that depends on x_k (recall that λ and μ are chosen such that $a_{\lambda\mu} \neq 0$, so that the latter Casimir does indeed depend on x_k). \square

When $n > m + 1$, there are other Poisson structures with respect to which (3.1) is Hamiltonian, with the same Hamiltonian $H = x_1 + \dots + x_m$. Indeed,

given any skew-symmetric $n \times n$ matrix (a_{ij}) , a Poisson bracket on \mathbb{C}^n is defined by $\{x_i, x_j\} := a_{ij}x_i x_j$ (see [11, Section 8.2]). Thus, given any $n \times m$ matrix A , whose upper square part is skew-symmetric, any extension of A to a skew-symmetric $n \times n$ matrix will lead to a Poisson structure on \mathbb{C}^n , making (3.1) a Hamiltonian vector field (with Hamiltonian H), and all these Poisson structures are compatible. In general, these Poisson structures are of rank higher than two; so, another choice of Poisson structure may have a negative impact on some of the above properties and some of the properties that we will establish next, such as Liouville integrability and a formulation in terms of Nambu-Poisson brackets. However, regarding superintegrability, solvability and Lax equations, the choice of Poisson structure does not play a rôle.

In the following proposition, we identify (3.1) as a Nambu system. Recall from [11, Section 8.3] that a Poisson structure of rank 2 on \mathbb{C}^n is obtained as follows: let C_3, C_4, \dots, C_n and χ be $n - 1$ functions on \mathbb{C}^n . For functions F, G on \mathbb{C}^n let

$$\{F, G\}_{NP} := \chi \left| \frac{\partial(F, G, C_3, \dots, C_n)}{\partial(x_1, \dots, x_n)} \right|. \quad (3.6)$$

This defines a Poisson structure on \mathbb{C}^n for which the given functions C_3, \dots, C_n are Casimir functions. It is called a *Nambu-Poisson structure*. Its rank is 2 at every point, except at the zeros of χ and the points $p \in \mathbb{C}^n$ where the differentials $d_p C_3, \dots, d_p C_n$ are linearly dependent (at those points the rank is zero).

Proposition 3.2. *Consider the Lotka-Volterra-Painlevé system (3.1), where we recall that at least one of the $a_{\lambda\mu}$ ($= \pi_{\lambda\mu}$) is non-zero, with $1 \leq \lambda, \mu \leq m$; in order to simplify the formulas, we assume that $a_{12} \neq 0$. Then the Poisson structure $\{\cdot, \cdot\}$ defined in Proposition 3.1 is a Nambu-Poisson structure, with Casimirs $C_k := x_1^{\pi_{2k}} x_2^{\pi_{k1}} x_k^{\pi_{12}}$ and with multiplier*

$$\chi := \frac{\prod_{k=1}^n x_k}{\pi_{12}^{n-3} \prod_{k=3}^n C_k}.$$

Proof. We need to check that $\{x_i, x_j\}_{NP} = \pi_{ij}x_i x_j$ for all $1 \leq i < j \leq n$. First, let $i = 1$ and $j = 2$. Then, according to (3.6), we find

$$\{x_1, x_2\}_{NP} = \chi \prod_{k=3}^n \frac{\partial C_k}{\partial x_k} = \chi \prod_{k=3}^n \left(\pi_{12} \frac{C_k}{x_k} \right) = \pi_{12} x_1 x_2.$$

For $j > 2$, one obtains

$$\{x_1, x_j\}_{NP} = -\chi \frac{\partial C_j}{\partial x_2} \prod_{k \neq j} \frac{\partial C_k}{\partial x_k} = -\chi \frac{\pi_{j1} C_j}{x_2} \prod_{k \neq j} \left(\frac{\pi_{12} C_k}{x_k} \right) = \pi_{1j} x_1 x_j,$$

and similarly, $\{x_2, x_j\}_{NP} = \pi_{2j}x_2x_j$. Finally, let $2 < i < j$. Then

$$\begin{aligned} \{x_i, x_j\}_{NP} &= \chi \prod_{k \neq i, j} \frac{\partial C_k}{\partial x_k} \left| \begin{array}{cc} \frac{\partial C_i}{\partial x_1} & \frac{\partial C_j}{\partial x_1} \\ \frac{\partial C_i}{\partial x_2} & \frac{\partial C_j}{\partial x_2} \end{array} \right| \\ &= \chi \prod_{k \neq i, j} \left(\frac{\pi_{12} C_k}{x_k} \right) \frac{C_i C_j}{x_1 x_2} (\pi_{2i} \pi_{j1} - \pi_{2j} \pi_{i1}) \\ &= \frac{x_i x_j}{\pi_{12}} (\pi_{2i} \pi_{j1} - \pi_{2j} \pi_{i1}) = \pi_{ij} x_i x_j, \end{aligned}$$

where we used in the last equality the relation (3.5). \square

Proposition 3.3. *Consider an arbitrary initial condition $x^{(0)} = (x_1^{(0)}, \dots, x_n^{(0)})$ for the Lotka-Volterra system*

$$\dot{x}_i = x_i \sum_{\lambda=1}^m a_{i\lambda} x_\lambda, \quad (i = 1, \dots, n), \quad (3.7)$$

and denote the value of the Hamiltonian H at $x^{(0)}$ by h , so $h := \sum_{\lambda=1}^m x_\lambda^{(0)}$. If $h \neq 0$ then the solution $x(t)$ of (3.7) with initial condition $x(0) = x^{(0)}$ is given by

$$x_\mu(t) = x_\mu^{(0)} \frac{h - x_\mu^{(0)}}{\sum_{\lambda \neq \mu} x_\lambda^{(0)} \exp(-a_{\mu\lambda} h t)}, \quad (i = 1, \dots, m), \quad (3.8)$$

$$x_i(t) = x_i^{(0)} \frac{h}{\sum_{\lambda=1}^m x_\lambda^{(0)} \exp(-a_{i\lambda} h t)}, \quad (i = m+1, \dots, n). \quad (3.9)$$

Otherwise, the solution is given by

$$x_i(t) = x_i^{(0)} \frac{1}{1 - t \sum_{\lambda \neq i} a_{i\lambda} x_\lambda^{(0)}}, \quad (i = 1, \dots, n). \quad (3.10)$$

Proof. When $x_i^{(0)} = 0$ the formulas (3.8) — (3.10) yield $x_i(t) = 0$, which is correct, so we only need to consider the case when $x_i^{(0)} \neq 0$. From (3.7) and using the cocycle condition we find for $\lambda = 1, \dots, m$ that

$$\frac{\dot{x}_\lambda}{x_\lambda} - \frac{\dot{x}_i}{x_i} = \sum_{\mu=1}^m (a_{\lambda\mu} - a_{i\mu}) x_\mu = -a_{i\lambda} \sum_{\mu=1}^m x_\mu = -a_{i\lambda} H. \quad (3.11)$$

By integration and upon using the initial condition,

$$x_\lambda(t) = \frac{x_\lambda^{(0)}}{x_i^{(0)}} x_i(t) \exp(-a_{i\lambda} h t).$$

Notice that this formula is correct even when $x_\lambda^{(0)} = 0$. Substituted in (3.7) we get

$$\frac{\dot{x}_i}{x_i^2}(t) = \sum_{\lambda=1}^m a_{i\lambda} \frac{x_\lambda^{(0)}}{x_i^{(0)}} \exp(-a_{i\lambda} h t). \quad (3.12)$$

When $h \neq 0$ a direct integration, taking into account the initial condition, yields

$$\frac{1}{x_i(t)} = \frac{1}{x_i^{(0)} \sum_{\lambda \neq i} x_\lambda^{(0)}} \sum_{\lambda \neq i} x_\lambda^{(0)} \exp(-a_{i\lambda} h t),$$

which is (3.8) or (3.9), depending on whether i is smaller or strictly larger than m . Otherwise, the right hand side of (3.12) is just a constant, to wit $\sum_{\lambda \neq i} a_{i\lambda} x_\lambda^{(0)} / x_i^{(0)}$, and we obtain at once (3.10). \square

The above proposition can be generalized in case the right hand side of the i -th equation of (3.7) has an additional linear term $b_i x_i$, where $b_i \in \mathbb{C}$. In that case, (3.11) becomes

$$\frac{\dot{x}_\lambda}{x_\lambda} - \frac{\dot{x}_i}{x_i} = -a_{i\lambda} H + b_\lambda - b_i,$$

which is a constant, as before, and so these equations can again be integrated in terms of elementary functions (it suffices to replace in the formulas $a_{i\lambda} h$ by $a_{i\lambda} h - b_\lambda + b_i$). However, the addition of these terms will in general destroy the Painlevé property (P1) (and hence (P2)), because the right hand side of the degenerate linear equations (2.13) gets replaced by $-b_i x_i^{(0)}$, making them non-linear, hence without solution, for general values of the constants b_i .

To conclude, we also give a Lax equation for the Lotka-Volterra-Painlevé systems.

Proposition 3.4. *The Lotka-Volterra-Painlevé system (3.1) admits the Lax form $\dot{L} = [L, M]$, where L and M are the rank one matrices (with m non-zero columns), defined by*

$$L := \begin{pmatrix} x_1 & \dots & x_1 & 0 & \dots & 0 \\ x_2 & \dots & x_2 & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ x_n & \dots & x_n & 0 & \dots & 0 \end{pmatrix}, \quad M := \begin{pmatrix} a_1 x_1 & \dots & a_1 x_1 & 0 & \dots & 0 \\ a_2 x_2 & \dots & a_2 x_2 & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ a_n x_n & \dots & a_n x_n & 0 & \dots & 0 \end{pmatrix}.$$

Proof. For any $i = 1, \dots, n$ and $\mu = 1, \dots, m$, we compute the (i, μ) -th entry of the Lax equation $\dot{L} = [L, M]$. On the one hand, $\dot{L}_{i\mu} = \dot{x}_i$. On the other hand,

$$[L, M]_{i\mu} = \sum_{\lambda=1}^m (L_{i\lambda} M_{\lambda\mu} - M_{i\lambda} L_{\lambda\mu}) = \sum_{\lambda=1}^m (a_\lambda x_i x_\lambda - a_i x_\lambda x_i) = x_i \sum_{\lambda=1}^m a_{i\lambda} x_\lambda,$$

so that the Lax equation is equivalent to $\dot{x}_i = x_i \sum_{\lambda=1}^m a_{i\lambda} x_\lambda$, ($i = 1, \dots, n$), which is precisely (3.1). \square

Notice that since L is of rank one, the only spectral invariant that we obtain from it is the trace of L , which is the Hamiltonian $H = \sum_{i=1}^n x_i$ of (3.1). As is often the case with Lax equations, we do not obtain the Casimirs as spectral invariants.

4. CONCLUSIONS

In this paper, we have studied Lotka–Volterra systems of the form

$$\dot{x}_i = x_i \sum_{\lambda=1}^m a_{i\lambda} x_\lambda, \quad (i = 1, \dots, n), \quad (4.1)$$

where $A = (a_{i\lambda})$ is an $n \times m$ skew-symmetric matrix derived from a Hamiltonian linear function $H = x_1 + x_2 + \dots + x_m$ through a family of compatible quadratic Poisson structures. By imposing on (4.1) what we call the “strong” Painlevé (P–) property, i.e that all variables can be expressed as Laurent series starting with a

simple pole and $n - 2$ free parameters at their leading order terms, we establish a set of cocycle relations through which we can integrate these systems completely.

More specifically, we have shown that, under the above conditions, the matrix elements $a_{i\lambda}$ can be written in terms of $n - 1$ free constants as $a_{i\lambda} = a_\lambda - a_i$, with $1 \leq i \leq n$, $1 \leq \lambda \leq m$ and can be used to demonstrate that the equations (4.1) are Liouville integrable, superintegrable, of Nambu type and can be written in Lax pair form. Moreover, they can be completely integrated in terms of elementary functions even if we add to the i -th equation a linear term of the form $\mu_i x_i$ with μ_i arbitrary and $1 \leq i \leq n$.

It is possible, of course, to consider Laurent series solutions of (4.1) with leading orders $x_i(t) \sim \tau^{p_i}$ other than simple poles, i.e. with $p_i > -1$ or $p_i < -1$ for some $1 \leq i \leq n$. Indeed, such systems are known, for example the so-called periodic Kac-van Moerbeke system, which corresponds in our notations to setting $a_{i,j} = \delta_{i+1,j}$ for $i < j$: it satisfies the Painlevé property and is actually algebraic completely integrable (see [9]). It would therefore be interesting to weaken our strong Painlevé property so as to capture also this class of systems. We plan to come back to this question in a future publication.

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