

# Integrable and superintegrable systems associated with multi-sums of products

Peter H. van der Kamp<sup>1</sup>, Theodoros E. Kouloukas<sup>1</sup>,  
G. R. W. Quispel<sup>1</sup>, Dinh T. Tran<sup>1</sup> and Pol Vanhaecke<sup>2</sup>

<sup>1</sup>Department of Mathematics, La Trobe University  
Bundoora (Melbourne) VIC 3086, Australia

<sup>2</sup>Laboratoire de Mathématiques et Applications,  
UMR 7348 du CNRS, Université de Poitiers,  
86962 Futuroscope Chasseneuil Cedex, France

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*Corresponding author:* peterhvanderkamp@gmail.com

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## Abstract

We construct and study certain Liouville integrable, superintegrable, and non-commutative integrable systems, which are associated with multi-sums of products.

## 1 Introduction

Integrable systems have a long and distinguished history. Starting with Newton's solution of the celestial two-body problem, the theory of integrable ordinary differential equations (ODEs) was put on a firm footing by Liouville. His theorem states that an autonomous Hamiltonian system in  $2m$  dimensions (or equivalently  $m$  degrees of freedom), that possesses  $m$  integrals in involution (i.e. whose mutual Poisson brackets all vanish) is integrable by quadrature. In the 1960's the discovery of solitons by Zabusky and Kruskal [27] heralded a major revival for integrable systems. It became clear that a significant number of partial differential equations (PDEs) was also to be regarded as integrable. Because a PDE may be considered to be an infinite set of coupled ODEs, they actually possess an infinite number of integrals in involution. The set of integrable PDEs includes several that have notable applications (e.g. the Korteweg-de Vries equation, the nonlinear Schrödinger equation and the sine-Gordon equation). For a survey of the theory of integrable ODEs and PDEs we refer the reader to [28, 14]. More recently *discrete* integrable systems have come to the fore. In these systems, all independent variables take on discrete values. All the above mentioned integrable PDEs have discrete analogues in the form of partial difference equations (PΔEs), which are integrable in their own right.

When imposing a periodicity condition a P $\Delta$ E reduces to an O $\Delta$ E or a mapping. For integrable P $\Delta$ Es the so-called staircase method yields a set of integrals for the reduced mapping [15, 22]. For maps obtained as reductions of the equations in the Adler-Bobenko-Suris (ABS) classification [1], for reductions of the sine-Gordon and modified Korteweg-De Vries (mKdV) equations, and for the  $p$ -th order Lyness equation, first integrals were given in closed form, in terms of multi-sums of products,  $\Psi$ , by using the staircase method and noncommutative Vieta expansion [18]. In particular, the Liouville integrability of mappings obtained as reductions of the discrete sine-Gordon, mKdV, pKdV, and KdV equations was studied in detail in [17, 20, 8].

This paper is concerned with integrable systems associated with a set of polynomials  $z_i^{(n)}$ , which are related to the multisums of products  $\Psi$ . The polynomials and a Poisson bracket  $\{\cdot, \cdot\}$  on  $\mathbb{R}^n$  are given in Section 2. We show that

$$\{z_{2k-1}^{(n)}, z_{2l-1}^{(n)}\} = \{z_{2k}^{(n)}, z_{2l}^{(n)}\} = 0 \quad (1.1)$$

for all  $k, l \in \{1, \dots, \lfloor \frac{n}{2} \rfloor\}$ , and therefore each polynomial  $z_i^{(n)}$  defines an associated integrable Hamiltonian vector field.<sup>1</sup> In Section 3, we consider the quadratic vector fields associated with  $z_3^{(n)}$ . This is an  $n$ -dimensional Lotka-Volterra system [26, 3], and we prove it is superintegrable when  $n$  is odd and non-commutative integrable (of rank 2) when  $n$  is even. We also apply the Kahan discretisation (sometimes also called Kahan-Hirota-Kimura discretisation [9, 10, 7]) to these quadratic vector fields, restricted to a subspace, and show that Liouville integrability and superintegrability are preserved.

## 2 Integrable systems associated to multi-sums of products

### 2.1 The polynomials $z_i$ and their independence

We introduce a set of  $n$  polynomial functions  $z_1, \dots, z_n$  on  $\mathbb{R}^n$ , and we show how they define two integrable systems on  $\mathbb{R}^n$ , with respect to a (constant) Poisson structure, which will be given below. The polynomials  $z_1, \dots, z_n$  are defined in terms of a large collection of polynomials  $\Upsilon_r^{a,b}$ , where  $a, b, r$  denote arbitrary integers, with  $r \geq 0$ .<sup>2</sup> For  $r > 0$ , the latter are defined in terms of  $(x_i)_{i \in \mathbb{Z}}$  by

$$\Upsilon_r^{a,b} := \sum_{\substack{a \leq i_1 < \dots < i_r \leq b, \\ i_j \equiv a+j-1 \pmod{2}}} \prod_{j=1}^r x_{i_j}. \quad (2.1)$$

Notice that  $\Upsilon_r^{a,b}$  is a homogeneous polynomial of degree  $r$  and that it depends only on the variables  $x_a, x_{a+1}, \dots, x_b$ ; in particular,

$$\Upsilon_r^{a,b} = 0 \quad \text{when } b - a + 1 < r. \quad (2.2)$$

<sup>1</sup>Some of these results were originally obtained in the PhD thesis of Dinh Tran [17].

<sup>2</sup>These polynomials relate to polynomials  $\Psi$  introduced in [18] by  $\Psi_r^{a,b} = \Upsilon_{b+1-a-2r}^{a,b+1}$ . They also are a special case of a much bigger class of polynomials which has been introduced in [24]. A different class of (non-polynomial) multi-sums of products,  $\theta$ , was introduced in [21]. For these multi-sums of products similar relations to (1.1) were derived in [20, Lemma 1].

Moreover,  $\Upsilon_r^{a,b}$  satisfies, for all  $a, b$  and for  $r > 1$ , the following two recursion relations

$$\Upsilon_r^{a,b} = \Upsilon_r^{a+2,b} + x_a \Upsilon_{r-1}^{a+1,b}, \quad \Upsilon_r^{a,b} = \Upsilon_r^{a,b-1} + \epsilon_r^{b-a} x_b \Upsilon_{r-1}^{a,b-1}, \quad (2.3)$$

where

$$\epsilon_r^q := \begin{cases} 0 & \text{if } q \text{ and } r \text{ have the same parity,} \\ 1 & \text{otherwise.} \end{cases}$$

In order for these recursion relations to make sense and be correct also for  $r = 1$ , the following additional definition is needed:

$$\Upsilon_0^{a,b} := \begin{cases} 0 & \text{if } a - 1 > b, \\ 1 & \text{if } a - 1 \leq b. \end{cases} \quad (2.4)$$

It is easy to see that the functions  $\Upsilon_r^{a,b}$  are uniquely determined by either one of the recursion relations in (2.3), together with (2.2) and (2.4). In terms of the functions  $\Upsilon_r^{a,b}$ , we define

$$z_i := \Upsilon_i^{1,n} \quad (2.5)$$

for  $i = 1, \dots, n$  and we view each  $z_i$  as a polynomial function on  $\mathbb{R}^n$ ; when  $n$  is not clear from the context, we also write  $z_i^{(n)}$  for  $z_i$ . It is also convenient to extend the definition (2.5) to arbitrary  $i \geq 0$ , by defining  $z_0 := 1$  and  $z_i := 0$  for  $i > n$ . In terms of the functions  $z_i$ , the second recursion relation in (2.3) leads to

$$z_i^{(n+1)} = z_i^{(n)} + \epsilon_i^n x_{n+1} z_{i-1}^{(n)}, \quad (2.6)$$

for any  $i > 0$ , i.e.,  $z_i^{(n+1)} = z_i^{(n)}$  if  $i$  and  $n$  have the same parity and  $z_i^{(n+1)} = z_i^{(n)} + x_{n+1} z_{i-1}^{(n)}$  otherwise. In particular,  $z_i^{(n)} = z_i^{(n+1)}|_{x_{n+1}=0}$  for all  $i$  and  $n$ . Also, each polynomial  $z_i$  is homogeneous and has degree  $i$ . Here are a few low-dimensional examples:

dimension 1 :  $z_1 = x_1$ ,

dimension 2 :  $z_1 = x_1$ ,  $z_2 = x_1 x_2$ ,

dimension 3 :  $z_1 = x_1 + x_3$ ,  $z_2 = x_1 x_2$ ,  $z_3 = x_1 x_2 x_3$ ,

dimension 4 :  $z_1 = x_1 + x_3$ ,  $z_2 = x_1 x_2 + x_1 x_4 + x_3 x_4$ ,  $z_3 = x_1 x_2 x_3$ ,  $z_4 = x_1 x_2 x_3 x_4$ ,

dimension 5 :  $z_1 = x_1 + x_3 + x_5$ ,  $z_2 = x_1 x_2 + x_1 x_4 + x_3 x_4$ ,  $z_3 = x_1 x_2 x_3 + x_1 x_2 x_5 + x_1 x_4 x_5$   
 $+ x_3 x_4 x_5$ ,  $z_4 = x_1 x_2 x_3 x_4$ ,  $z_5 = x_1 x_2 x_3 x_4 x_5$ .

**Proposition 2.1.** *On the open dense subset  $D$  of  $\mathbb{R}^n$ , defined by*

$$D := \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_1 x_2 \dots x_{n-1} \neq 0\} \quad (2.7)$$

*the functions  $z_1, \dots, z_n$  have independent differentials, hence they define a coordinate system on a neighborhood of any point of  $D$ .*

*Proof.* We need to show that at every point of  $D$  the rank of the Jacobian matrix

$$J^{(n)} := \frac{\partial(z_1, \dots, z_n)}{\partial(x_1, x_2, \dots, x_n)} = \left( \frac{\partial z_i^{(n)}}{\partial x_j} \right)_{ij} \quad (2.8)$$

is equal to  $n$ . To do this, we use an LU-decomposition of  $J^{(n)}$ , i.e. we write  $J^{(n)} = L^{(n)}U^{(n)}$ , where  $L^{(n)}$  is a lower triangular matrix and  $U^{(n)}$  is an upper triangular matrix; we show that all entries on the diagonal of  $L^{(n)}$  and of  $U^{(n)}$  are non-zero at every point of  $D$ . Precisely, we show that the upper triangular entries of  $L^{(n)}$  are given by  $L_{ij}^{(n)} = \Upsilon_{i-j}^{j+1,n}$ , so that all diagonal entries of  $L^{(n)}$  are equal to 1, and that the diagonal entries of  $U^{(n)}$  are given by  $U_{kk}^{(n)} = x_1x_2 \dots x_{k-1}$ . We do this by induction. For  $n = 1$  it is clear, so let us assume that  $J^{(n)} = L^{(n)}U^{(n)}$  for some  $n > 0$ , with  $L^{(n)}$  and  $U^{(n)}$  as above. As before, we usually drop the superscript  $n$  and simply write  $J = LU$ . We need to prove that we can write  $J^{(n+1)} = L^{(n+1)}U^{(n+1)}$ , with  $L^{(n+1)}$  lower triangular,  $U^{(n+1)}$  upper triangular, with entries as given above. Using the recursion relation (2.6), we can write  $J^{(n+1)}$  in terms of  $J$ , to wit,

$$J^{(n+1)} = \begin{pmatrix} J & \mathbf{0} \\ \mathbf{0} & 0 \end{pmatrix} + \begin{pmatrix} \mathbf{0} & 0 \\ \bar{J} & \mathbf{0} \end{pmatrix} + \begin{pmatrix} 0 & 0 & \dots & 0 & \epsilon_1^n \\ 0 & 0 & \dots & 0 & \epsilon_2^n z_1 \\ 0 & 0 & \dots & 0 & \epsilon_3^n z_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \epsilon_{n+1}^n z_n \end{pmatrix}, \quad (2.9)$$

where  $\bar{J}$  is the  $(n \times n)$ -matrix obtained from  $J$  by multiplying its  $k^{\text{th}}$  row by  $\epsilon_{k+1}^n x_{n+1}$ , for  $k = 1, \dots, n$ . We similarly define  $\bar{L}$ , starting from  $L$ . Then the relation  $J = LU$  implies  $\bar{J} = \bar{L}U$ , so that the first two terms in (2.9) can be written as

$$\left( \begin{pmatrix} L & \mathbf{0} \\ \mathbf{0} & 0 \end{pmatrix} + \begin{pmatrix} \mathbf{0} & 0 \\ \bar{L} & \mathbf{0} \end{pmatrix} \right) \begin{pmatrix} U & \mathbf{0} \\ \mathbf{0} & 0 \end{pmatrix} = L^{(n+1)} \begin{pmatrix} U & \mathbf{0} \\ \mathbf{0} & 0 \end{pmatrix}, \quad (2.10)$$

where  $L^{(n+1)}$  can be chosen as a lower triangular matrix:  $L$  itself is lower triangular, with 1's on the diagonal, and we choose the last diagonal entry of  $L^{(n+1)}$  also equal to 1. For  $1 \leq j < i \leq n$  we have, using (2.3) and the induction hypothesis,

$$L_{ij}^{(n+1)} = L_{ij} + \bar{L}_{i-1,j} = \Upsilon_{i-j}^{j+1,n} + \epsilon_i^n x_{n+1} \Upsilon_{i-j-1}^{j+1,n} = \Upsilon_{i-j}^{j+1,n+1}; \quad (2.11)$$

similarly, if  $i = n+1$  and  $j < n$ , then

$$L_{n+1,j}^{(n+1)} = \bar{L}_{nj} = x_{n+1} \Upsilon_{n-j}^{j+1,n} = \Upsilon_{n-j+1}^{j+1,n+1}.$$

This shows that  $L^{(n+1)}$  and its entries have the asserted form. We now turn our attention to  $U^{(n+1)}$ . Let us denote the last column of the last term in (2.9) by  $\mathbf{j}$ , so  $\mathbf{j}_i = \epsilon_i^n z_{i-1}$ , for  $i = 1, \dots, n+1$ . Since  $L^{(n+1)}$  is invertible there exists a unique column vector  $\mathbf{u}$  such that  $L^{(n+1)}\mathbf{u} = \mathbf{j}$ . It leads to the LU decomposition  $J^{(n+1)} = L^{(n+1)}U^{(n+1)}$ , where  $U^{(n+1)}$  is the upper triangular matrix, defined by

$$U^{(n+1)} := \begin{pmatrix} U & \mathbf{0} \\ \mathbf{0} & 0 \end{pmatrix} + \begin{pmatrix} \mathbf{0} & \mathbf{u} \end{pmatrix}.$$

In order to prove that the entries of  $U^{(n+1)}$  have the asserted form, it suffices to show that  $\mathbf{u}_{n+1} = x_1x_2 \dots x_n$ . If we denote the last row of  $(L^{(n+1)})^{-1}$  by  $\mathbf{t}$ , then  $\mathbf{u}_{n+1} = \mathbf{t}\mathbf{j}$ ; we will show that  $\mathbf{t} = (0, \dots, 0, -x_{n+1}, 1)$ , from which we obtain

$$\mathbf{u}_{n+1} = \mathbf{t}\mathbf{j} = -\epsilon_n^n x_{n+1} z_{n-1} + \epsilon_{n+1}^n z_n = z_n = x_1x_2 \dots x_n,$$

as was to be shown. In order to prove the proposed formula for  $\mathbf{t}$  it suffices to show that  $(0, \dots, 0, -x_{n+1}, 1)L^{(n+1)} = (0, 0, \dots, 0, 1)$ , which is tantamount to saying that  $L_{n+1, n+1}^{(n+1)} = 1$  (which is true by definition) and that  $L_{n+1, k}^{(n+1)} - x_{n+1}L_{n, k}^{(n+1)} = 0$  for  $k = 1, \dots, n$ . In terms of the matrices  $L$  and  $\bar{L}$  this amounts to  $\bar{L}_{nk} = x_{n+1}L_{nk}$ , which is precisely the definition of the last row of  $\bar{L}$ .  $\square$

*Remark 2.2.* The map  $x \rightarrow z$  is in fact a birational map, i.e., one can write  $x_1, \dots, x_n$  as rational functions of  $z_1, \dots, z_n$  (while each  $z_i$  is a polynomial in the  $x_k$ ). The proof goes by induction on  $n$ . Suppose that we have shown that  $x_1, \dots, x_n$  can be written as rational functions of  $z_1^{(1)}, \dots, z_n^{(n)}$ ,

$$x_k = R_k(z_1^{(n)}, \dots, z_n^{(n)}) = R_k \left( z_i^{(n)} \right)_{i=1, \dots, n},$$

where the latter notation will come in handy soon. Repeated use of (2.6) leads to

$$\begin{aligned} z_{n+1}^{(n+1)} &= x_{n+1}z_n^{(n)} = \dots = x_1x_2 \dots x_{n+1}, \\ z_n^{(n+1)} &= z_n^{(n)} = x_1x_2 \dots x_n, \end{aligned}$$

so that

$$x_{n+1} = z_{n+1}^{(n+1)} / z_n^{(n+1)}. \quad (2.12)$$

Substituted in (2.6) we get, for  $i = 1, \dots, n$ ,

$$z_i^{(n)} = z_i^{(n+1)} - \epsilon_i^n \frac{z_{n+1}^{(n+1)}}{z_n^{(n+1)}} z_{i-1}^{(n)} = z_i^{(n+1)} - \epsilon_i^n \frac{z_{n+1}^{(n+1)}}{z_n^{(n+1)}} z_{i-1}^{(n+1)},$$

where we used in the last step that if  $\epsilon_i^n \neq 0$  (so that  $n$  and  $i$  have opposite parity), then  $z_{i-1}^{(n+1)} = z_{i-1}^{(n)}$ . It follows that

$$x_k = R_k \left( z_i^{(n+1)} - \epsilon_i^n \frac{z_{n+1}^{(n+1)}}{z_n^{(n+1)}} z_{i-1}^{(n+1)} \right)_{i=1, \dots, n}, \quad (2.13)$$

for  $k = 1, \dots, n$ , Equations (2.12) and (2.13), together, show that  $x_1, \dots, x_{n+1}$  can be written as rational functions of  $z_1^{(1)}, \dots, z_{n+1}^{(n+1)}$ . Combined with Proposition 2.1 it shows that  $z_1, \dots, z_n$  form a coordinate system on  $D$ .

## 2.2 The Poisson structure and involutivity

On  $\mathbb{R}^n$  we consider the constant Poisson structure, defined by

$$\{x_i, x_j\}_n := \delta_{i+1, j} - \delta_{i, j+1}. \quad (2.14)$$

When  $n$  is even, its rank is  $n$ ; otherwise its rank is  $n - 1$  and  $z_1 = \sum_{i=1}^{(n+1)/2} x_{2i-1}$  is a Casimir function, since  $\{x_i, z_1\} = 0$  for all  $i$ . In geometrical terms, the Poisson structure in the odd-dimensional case is obtained by a Poisson reduction (see [12, Ch. 5.2]) from the Poisson structure in the even-dimensional case: if we view  $\mathbb{R}^{2m-1}$  as the quotient of  $\mathbb{R}^{2m}$  under the quotient map  $\pi : \mathbb{R}^{2m} \rightarrow \mathbb{R}^{2m-1}$ , defined by  $(x_1, x_2, \dots, x_{2m}) \mapsto (x_1, x_2, \dots, x_{2m-1})$ , then

the pair  $(\mathbb{R}^{2m}, \mathbb{R}^{2m-1})$  is Poisson reducible and the reduced Poisson structure, inherited from  $\{\cdot, \cdot\}_{2m}$  is  $\{\cdot, \cdot\}_{2m-1}$ . In particular,  $\pi : (\mathbb{R}^{2m}, \{\cdot, \cdot\}_{2m}) \rightarrow (\mathbb{R}^{2m-1}, \{\cdot, \cdot\}_{2m-1})$  is a Poisson map.

In the following proposition we give explicit formulas for the Poisson brackets between the functions  $z_1, \dots, z_n$ . We write here, and in the rest of the paper,  $a \equiv b$  when  $a \equiv b \pmod{2}$ , i.e., when the integers  $a$  and  $b$  have the same parity.

**Proposition 2.3.** *For any  $i$  and  $j$  with  $0 \leq i, j \leq n$ , we have that*

$$\{z_i, z_j\}_n = \begin{cases} 0 & \text{if } i \equiv j, \\ \sum_{\substack{k+\ell=i+j \\ 0 < k < i}} (-1)^{i-k} z_{k-1} z_{\ell-1} & \text{if } i \equiv j+1 \equiv n. \end{cases}$$

*Proof.* The proof proceeds by induction on  $n$ . It is easy to check that the formulas hold for  $n = 1$ . Suppose that they hold for some  $n \geq 1$ . We need to prove that they hold for  $n + 1$ . Let  $0 \leq i, j \leq n + 1$  be arbitrary integers. If  $i = 0$  or  $j = 0$  the formula is easily checked (recall that  $z_0 = 1$ ), so let us assume that  $i, j > 0$ . Using (2.6), we have

$$\left\{ z_i^{(n+1)}, z_j^{(n+1)} \right\}_{n+1} = \left\{ z_i^{(n)} + \epsilon_i^n x_{n+1} z_{i-1}^{(n)}, z_j^{(n)} + \epsilon_j^n x_{n+1} z_{j-1}^{(n)} \right\}_{n+1}. \quad (2.15)$$

We use  $\dot{F}$  as a shorthand for  $\{F, x_{n+1}\}_{n+1} = \partial F / \partial x_n$  and we write  $z_k$  for  $z_k^{(n)}$  for any  $k$ . We have that  $\{z_i, z_j\}_{n+1} = \{z_i, z_j\}_n$  because  $z_i$  and  $z_j$  depend on  $x_1, \dots, x_n$  only. We distinguish 3 cases, according to the relative parity of  $i, j$  and  $n$ .

•  $i \equiv j \equiv n$  Then  $\epsilon_i^n = \epsilon_j^n = 0$  and  $j < n + 1$ , so that (2.15) becomes

$$\left\{ z_i^{(n+1)}, z_j^{(n+1)} \right\}_{n+1} = \{z_i, z_j\}_n = 0.$$

•  $i \equiv j \equiv n + 1$  Then  $\epsilon_i^n = \epsilon_j^n = 1$ . We expand the right hand side of (2.15) and use that  $\{z_i, z_j\} = 0$  and that  $\dot{z}_i = \dot{z}_j = 0$  ( $z_i$  and  $z_j$  are independent of  $x_n$  since  $i \equiv j \equiv n + 1$ ) to obtain

$$\left\{ z_i^{(n+1)}, z_j^{(n+1)} \right\}_{n+1} = x_{n+1} (\{z_i, z_{j-1}\}_n + \dot{z}_{i-1} z_{j-1} - \{z_j, z_{i-1}\}_n - \dot{z}_{j-1} z_{i-1}).$$

Since  $i$  and  $n$  have opposite parity, we find from (2.6) that

$$\dot{z}_{i-1} = \frac{\partial z_{i-1}^{(n)}}{\partial x_n} = z_{i-2}^{(n-1)} = z_{i-2}^{(n)} = z_{i-2},$$

and similarly for  $\dot{z}_{j-1}$ , so it suffices to show that

$$\{z_{i-1}, z_j\}_n + z_{i-2} z_{j-1} - \{z_{j-1}, z_i\}_n - z_{j-2} z_{i-1} = 0. \quad (2.16)$$

Using the induction hypothesis, the left hand side in (2.16) is given by

$$\sum_{\substack{k+\ell=i+j-1 \\ 0 < k < i}} (-1)^{i-1-k} z_{k-1} z_{\ell-1} - \sum_{\substack{k+\ell=i+j-1 \\ 0 < k < j}} (-1)^{j-1-k} z_{k-1} z_{\ell-1}$$

which is zero, because every term appears twice with opposite signs (recall that  $i \equiv j$ ).

•  $i \equiv j + 1$  By interchanging  $i$  and  $j$  if needed, we may suppose that  $j \equiv n$ . Then  $\epsilon_i^n = 1$  and  $\epsilon_j^n = 0$  so that we get, as above,

$$\left\{ z_i^{(n+1)}, z_j^{(n+1)} \right\}_{n+1} = \{z_i, z_j\}_n - z_{i-1} \dot{z}_j = \sum_{\substack{k+\ell=i+j \\ 0 < k \leq j}} (-1)^{j-k-1} z_{k-1} z_{\ell-1}.$$

This is to be compared with

$$\begin{aligned} & \sum_{\substack{k+\ell=i+j \\ 0 < k < i}} (-1)^{i-k} z_{k-1}^{(n+1)} z_{\ell-1}^{(n+1)} \\ &= \sum_{\substack{k+\ell=i+j \\ 0 < k < i}} (-1)^{i-k} (z_{k-1} + \epsilon_{k-1}^n x_{n+1} z_{k-2}) (z_{\ell-1} + \epsilon_{\ell-1}^n x_{n+1} z_{\ell-2}) \\ &= \sum_{\substack{k+\ell=i+j \\ 0 < k < i}} (-1)^{i-k} z_{k-1} z_{\ell-1} + x_{n+1} \left[ \sum_{\substack{k+\ell=i+j \\ 0 < k < i-1 \\ k \equiv i}} z_{k-1} z_{\ell-2} - \sum_{\substack{k+\ell=i+j \\ 1 < k < i \\ k \equiv i+1}} z_{k-2} z_{\ell-1} \right] \\ &= \sum_{\substack{k+\ell=i+j \\ 0 < k < i}} (-1)^{i-k} z_{k-1} z_{\ell-1}. \end{aligned}$$

Taking the difference of both expressions we get zero, again because in the difference every term appears twice with opposite signs.  $\square$

### 2.3 The $\Upsilon$ -systems, their Liouville integrability and linearisation

Recall the definition of a Liouville integrable system:

**Definition 2.4.** On an  $(n = 2r + s)$ -dimensional Poisson manifold where the Poisson bracket has rank  $2r$ , a tuple of  $n - r = r + s$  functionally independent functions is *Liouville integrable* if they are (pairwise) in involution.

According to Proposition 2.3, the functions  $z_i$  with even (resp. odd) index  $i$  are pairwise in involution. Since for  $n$  odd the function  $z_1$  is a Casimir, hence is in involution with all functions  $z_i$ , we define

$$\begin{aligned} \mathbf{F} &:= (z_1, z_3, \dots, z_{n-1}), & \mathbf{F}' &:= (z_2, z_4, \dots, z_n) && \text{if } n \text{ is even,} \\ \mathbf{F} &:= (z_1, z_3, \dots, z_n), & \mathbf{F}' &:= (z_1, z_2, z_4, \dots, z_{n-1}) && \text{if } n \text{ is odd.} \end{aligned}$$

We use the above results to show in the following theorem that both  $\mathbf{F}$  and  $\mathbf{F}'$  are integrable systems on  $(\mathbb{R}^n, \{\cdot, \cdot\}_n)$ .

**Theorem 2.5.** *Both  $\mathbf{F}$  and  $\mathbf{F}'$  are Liouville integrable systems on  $(\mathbb{R}^n, \{\cdot, \cdot\}_n)$ . On the open subset  $D$ , defined in Proposition 2.1, the functions  $z_1, \dots, z_n$  define a coordinate system in terms of which all the Hamiltonian vector fields  $\mathcal{X}_{z_i}$  are linear.*

*Proof.* Recall that the rank of the Poisson structure  $\{\cdot, \cdot\}_n$  is equal to  $n$  when  $n$  is even, and is otherwise equal to  $n - 1$ . When  $n$  is even both  $\mathbf{F}$  and  $\mathbf{F}'$  contain  $n/2$  functions which

are independent (according to Proposition 2.1) and are pairwise in involution (Proposition 2.3). Thus, both  $\mathbf{F}$  and  $\mathbf{F}'$  define Liouville integrable systems on  $(\mathbb{R}^n, \{\cdot, \cdot\}_n)$ . When  $n$  is odd the rank of the Poisson structure is  $n - 1$  and so we need, besides the Casimir  $z_1$ , another  $(n - 1)/2$  independent functions in involution. Using Propositions 2.1 and 2.3 we can again conclude that both  $\mathbf{F}$  and  $\mathbf{F}'$  define Liouville integrable systems on  $(\mathbb{R}^n, \{\cdot, \cdot\}_n)$ .

In terms of the coordinates  $z_1, \dots, z_n$  on  $D$ , the Hamiltonian vector fields  $\mathcal{X}_{z_i} := \{\cdot, z_i\}_n$  take a particularly simple form. We show this for even  $n$ . According to Proposition 2.3, these vector fields are given by:

$$\begin{aligned} \mathcal{X}_{z_{2s}} &: \begin{cases} \dot{z}_{2r} &= 0, \\ \dot{z}_{2r-1} &= \sum_{i=1}^{2s-1} (-1)^{i-1} z_{2s-i-1} z_{2r+i-2}, \end{cases} \\ \mathcal{X}_{z_{2s-1}} &: \begin{cases} \dot{z}_{2r-1} &= 0, \\ \dot{z}_{2r} &= \sum_{i=1}^{2r-1} (-1)^i z_{2r-i-1} z_{2s+i-2}, \end{cases} \end{aligned}$$

where the dot denotes differentiation, i.e.  $\dot{z} = \frac{dz}{dt}$ . Each one of these vector fields becomes a linear vector field after the first group of equations has been integrated (giving  $z_{2r} = c_{2r}$  for  $\mathcal{X}_{z_{2s}}$  and  $z_{2r-1} = c_{2r-1}$  for  $\mathcal{X}_{z_{2s-1}}$ , where the  $c_i$  are constants).  $\square$

In the sequel, we refer to the integrable systems  $(\mathbb{R}^n, \{\cdot, \cdot\}_n, \mathbf{F})$  (resp. the integrable systems  $(\mathbb{R}^n, \{\cdot, \cdot\}_n, \mathbf{F}')$ ) as the *odd* (resp. the *even*)  $\Upsilon$ -systems in dimension  $n$ .

## 2.4 Lax equations for the $\Upsilon$ -systems

In this subsection, we show that the integrable vector fields  $\mathcal{X}_{z_i}$  of the  $\Upsilon$ -systems, defined in Section 2.3, are given by Lax equations, i.e. for each integrable system (in both odd and even dimensions) we provide a matrix  $L$  and matrices  $B_i$ , such that each vector field of the system acts on  $L$  as the commutator with  $B_i$ :

$$\mathcal{X}_{z_i}(L) = [L, B_i].$$

This yields an alternative proof (i.e., without using Proposition 2.3) that the functions  $z_i$  and  $z_j$  are in involution, when  $i \equiv j$ . We first consider the case of the even  $\Upsilon$ -systems in the even-dimensional case ( $n$  even) and derive from it the odd-dimensional case ( $n$  odd); for the case of the odd  $\Upsilon$ -system we first treat the odd-dimensional case and derive then the even-dimensional case from it.

The case of  $\mathbf{F}'$  with  $n$  even. We set  $n = 2m$ . We show that a Lax operator for  $\mathbf{F}'$  on  $\mathbb{R}^n$  is given by the following  $n \times n$  matrix:

$$L'_{2m} = \begin{pmatrix} 0 & x_1 & 0 & x_1 & 0 & x_1 & \cdots & 0 & x_1 \\ -x_2 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & x_3 & 0 & x_3 & \cdots & 0 & x_3 \\ -x_4 & 0 & -x_4 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & x_{2m-1} \\ -x_{2m} & 0 & -x_{2m} & 0 & -x_{2m} & 0 & \cdots & -x_{2m} & 0 \end{pmatrix}. \quad (2.17)$$

To do this, we first show that the polynomials in  $\mathbf{F}' = (z_2, z_4, \dots, z_{2m})$  appear as coefficients of the characteristic polynomial of  $L'_{2m}$ . First notice that it follows at once from a Laplace



expansion of the determinant with respect to the last two rows that  $\det(L'_{2m}) = x_1 x_2 \dots x_n$ . For any  $s$ , we define a tridiagonal matrix  $M'_s$  by setting  $(M'_s)_{i,j} := x_j^{-1}(\delta_{i,j+1} - \delta_{j,i+1})$ . It is easy to verify that  $M'_{2m}$  is the inverse of  $L'_{2m}$ . Thus,

$$|L'_{2m} - \lambda I_{2m}| = |L'_{2m}| |I_{2m} - l M'_{2m}| = l^{2m} |L'_{2m}| |M'_{2m} - \frac{1}{l} I_{2m}|, \quad (2.18)$$

and it suffices to compute the characteristic polynomial of  $M'_{2m}$ . We claim that for any  $s$ ,

$$p'_s(\mu) := |M'_s - \mu I_s| = \frac{(-1)^s}{z_s^{(s)}} \sum_{k=0}^{\lfloor s/2 \rfloor} z_{s-2k}^{(s)} \mu^{s-2k}. \quad (2.19)$$

Notice that the formula for  $p'_s$  is obviously correct for  $s = 1$  and  $s = 2$ . Expanding  $|M'_s - \mu I_s|$  along its last column, one finds that

$$p'_{s+2}(\mu) = \frac{p'_s(\mu)}{x_{s+1} x_{s+2}} - \mu p'_{s+1}(\mu) \quad (2.20)$$

for all  $s \geq 1$ , so it suffices to show that the formula for  $p'_s(\mu)$ , given in (2.19) satisfies the recursion relation (2.20). That this is indeed so follows easily from (2.6), combined with the formula  $z_s^{(s)} = x_1 x_2 \dots x_s$ . Combined with (2.18), this proves that the characteristic polynomial of  $L'_{2m}$  is given by

$$|L'_{2m} - \lambda I_{2m}| = \sum_{i=0}^m z_{2i} l^{2m-2i};$$

in particular, its coefficients are precisely the Hamiltonians  $z_2, z_4, \dots, z_n$  which make up  $\mathbf{F}'$ . For  $k = 1, \dots, m$ , a matrix  $B'_{2m,2k}$  satisfying

$$\mathcal{X}_{z_{2k}}(L'_{2m}) = [L'_{2m}, B'_{2m,2k}], \quad (2.21)$$

can be chosen upper triangular and with entries  $b_{ij}$  given by

$$b_{ij} := \begin{cases} \sum_{\substack{0 \leq r \leq 2k-2 \\ r \equiv i}} (\Upsilon_r^{1,i} \delta_{ij} - x_i \Upsilon_{r-1}^{1,i-1}) \Upsilon_{2k-2-r}^{j+1,n} & \text{if } i \leq j \text{ and } i \equiv j \\ 0 & \text{otherwise.} \end{cases} \quad (2.22)$$

We will give a proof that the matrices  $L'_{2m}$  and  $B'_{2m,2k}$  satisfy (2.21) in Appendix C.

The case of  $\mathbf{F}'$  with  $n$  odd. We set  $n = 2m - 1$  and construct a Lax operator for  $\mathbf{F}'$  (in dimension  $2m - 1$ ), by slightly modifying the matrix  $L'_{2m}$  (which has size  $2m$ ): we substitute 0 for  $x_{2m}$  in  $L'_{2m}$  (making all entries on its last row equal to zero) and for all entries in the last column except the second-to-last entry (which is equal to  $x_{2m-1}$ ). This yields a Lax operator  $L'_{2m-1}$  for  $\mathbf{F}'$  in the odd dimension  $2m - 1$ . Its characteristic polynomial is given by

$$|L'_{2m-1} - l I_{2m}| = l^2 |L'_{2m-2} - l I_{2m}| = \sum_{i=0}^{m-1} z_{2i}^{(2m-2)} l^{2m-2i} = \sum_{i=0}^{m-1} z_{2i}^{(2m-1)} l^{2m-2i}, \quad (2.23)$$

where we used that  $z_{2i}^{(2m-2)} = z_{2i}^{(2m-1)}$ , an immediate consequence of (2.6). This shows that, except for the Casimir  $z_1$ , all functions in  $\mathbf{F}' = (z_1, z_2, z_4, \dots, z_{n-1})$  appear as coefficients of

the characteristic polynomial of  $L'_{2m-1}$ . For  $k = 1, \dots, m-1$ , the matrix  $B'_{2m-1,2k}$  is obtained from  $B'_{2m,2k}$  as follows: replace all entries in its last two rows and in its last two columns by 0, except for the entry at position  $(2m-1, 2m-1)$ , which is set equal to  $z_{2k-1}^{(2m-3)}/x_{2m-1}$ . For a proof that the matrices  $L'_{2m-1}$  and  $B'_{2m-1,2k}$  satisfy a Lax equation as in (2.21), we refer again to Appendix C.

The case of  $\mathbf{F}$  with  $n$  odd. We set, as before,  $n = 2m - 1$ . A Lax operator for  $\mathbf{F} = (z_1, z_3, \dots, z_n)$  on  $\mathbb{R}^n$  is given by the following  $n \times n$  matrix:

$$L_{2m-1} = \begin{pmatrix} x_1 & x_1 & x_1 & x_1 & x_1 & \cdots & x_1 & x_1 \\ 0 & 0 & x_2 & 0 & x_2 & \cdots & 0 & x_2 \\ x_3 & 0 & x_3 & x_3 & x_3 & \cdots & x_3 & x_3 \\ 0 & 0 & 0 & 0 & x_4 & \cdots & 0 & x_4 \\ x_5 & 0 & x_5 & 0 & x_5 & \cdots & x_5 & x_5 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ x_{2m-1} & 0 & x_{2m-1} & 0 & x_{2m-1} & \cdots & 0 & x_{2m-1} \end{pmatrix} \quad (2.24)$$

It can be shown by induction that  $\det L_{2m-1} = x_1 x_2 \dots x_{2m-1}$ . For any  $s$ , consider the matrix  $M_s$ , which is obtained from the matrix  $M'_s$  by replacing the entry in its upper right corner (which is zero) by  $x_{2m-1}^{-1}$ . It is easy to verify that  $M_{2m-1}$  is the inverse of  $L_{2m-1}$ . Expanding  $|M_s - \mu I_s|$  along its last column, we find that  $p_s(\mu) = p'_s(\mu) + (x_1 x_2 \dots x_s)^{-1}$ . Using the explicit formula (2.19) for  $p'_s(\mu)$  it follows, as in (2.18), that

$$|L_{2m-1} - \lambda I_{2m-1}| = -l^{2m-1} + \sum_{i=1}^m z_{2i-1} l^{2m-2i} ;$$

in particular, its coefficients are precisely the Hamiltonians  $z_1, z_3, \dots, z_n$  which make up  $\mathbf{F}$ . The matrix  $B_{2m-1,2k-1}$  is defined as in (2.22), but with  $2k$  replaced by  $2k-1$ . Notice that this yields that  $B_{2m-1,1} = 0$ , which is correct since  $z_1$  is a Casimir. The proof that

$$\mathcal{X}_{z_{2k-1}}(L_{2m-1}) = [L_{2m-1}, B_{2m-1,2k-1}] ,$$

for  $k = 2, \dots, m$  is an easy adaptation of the proof given in Appendix C.

The case of  $\mathbf{F}$  with  $n$  even. We set  $n = 2m$  and we construct the Lax operator  $L_{2m}$  from  $L_{2m+1}$  by substituting in it 0 for  $x_{2m+1}$  (making all entries on its last row equal to zero) and for all entries in the last column except the second-to-last entry (which is equal to  $x_{2m}$ ). One obtains, as in (2.23),

$$|L_{2m} - l I_{2m+1}| = l^2 |L_{2m-1} - l I_{2m-1}| = -l^{2m+1} + \sum_{i=1}^m z_{2i-1}^{(2m)} l^{2m+2-2i} .$$

The matrix  $B_{2m,2k-1}$  is obtained from the matrix  $B_{2m+1,2k-1}$  as follows: replace all entries in its last two rows and in its last two columns by 0, except for the following entries

$$b_{1,2m} = -z_{2k-2}^{(2m)}/x_{2m}, \quad b_{2m,2m} = z_{2k-2}^{(2m)}/x_{2m}, \quad b_{2,2m+1} = -z_{2k-2}^{(2m)}/x_1 .$$

More details can be found in Appendix C.

### 3 On the quadratic vector fields

The integrable vector fields of the  $\Upsilon$ -systems are homogeneous, namely the Hamiltonian vector field associated to  $z_i$  is homogeneous of degree  $i - 1$ , since  $z_i$  is homogeneous of degree  $i$  (recall that the Poisson structure is constant). In this section we study the quadratic vector fields of the  $n$ -dimensional  $\Upsilon$ -systems. We show that they are closely related to a class of Lotka-Volterra systems in dimension  $m$ , where  $m = \lfloor (n + 1)/2 \rfloor$ . We establish the super- and Liouville integrability of these Lotka-Volterra systems, by exhibiting explicit rational constants of motion; notice that the only non-trivial dimension in which the integrability of these Lotka-Volterra systems was known is in dimension 4, see [6, Example 12]. We derive from it the superintegrability (if the dimension  $n$  is odd) and non-commutative integrability of rank 2 (if the dimension  $n$  is even) of the quadratic vector fields; recall that their Liouville integrability was obtained in the previous section. For the reader who is not familiar with these notions, here are the definitions.

**Definition 3.1.** A vector field on a  $n$ -dimensional manifold is *superintegrable* if it admits  $n - 1$  functionally independent constants of motion.

In particular, a Hamiltonian vector field is superintegrable if there is a functionally independent set of  $n - 1$  functions, including the Hamiltonian, which are in involution with the Hamiltonian.

**Definition 3.2.** On an  $n = 2r + s$ -dimensional Poisson manifold  $M$  where the Poisson bracket has rank  $\geq 2r$ , a tuple of  $n - r = r + s$  functionally independent functions is *non-commutative integrable of rank  $r$* , if  $r$  functions are in involution with all  $n - r$  functions, and their Hamiltonian vector fields are linearly independent at some point of  $M$ .

Thus, on a Poisson manifold, superintegrability is equivalent to non-commutative integrability of rank 1, and Liouville integrability is equivalent to non-commutative integrability of rank  $r$ , although here we would rather say that the system is commutative integrable. In both cases the rank of the system is at most half the rank of the Poisson bracket, with equality in the commutative (Liouville) case.

At the end of the section, we also derive from the superintegrability of the special Lotka-Volterra subsystems the superintegrability of their Kahan discretization; surprisingly, both the original system and the discretization have the same constants of motion.

#### 3.1 The quadratic vector fields

Recall that  $z_3$ , which is the Hamiltonian of the quadratic vector fields of the  $\Upsilon$ -systems, is given by

$$z_3 = \Upsilon_3^{1,n} = \sum_{\substack{1 \leq i < j < k \leq n \\ i, k \text{ odd}; j \text{ even}}} x_i x_j x_k .$$

When  $n$  is odd,  $n = 2m - 1$ , the quadratic vector field  $\mathcal{X}_{z_3}$  is explicitly given by

$$\begin{aligned} \dot{x}_{2\ell-1} &= x_{2\ell-1} \left( - \sum_{j=1}^{\ell-1} x_{2j-1} + \sum_{j=\ell+1}^m x_{2j-1} \right) \quad (\ell = 1, \dots, m), \\ \dot{x}_{2\ell} &= x_{2\ell} \left( \sum_{j=1}^{\ell} x_{2j-1} - \sum_{j=\ell+1}^m x_{2j-1} \right) \quad (\ell = 1, \dots, m-1); \end{aligned} \tag{3.1}$$

when  $n$  is even,  $n = 2m$ , it is given by (3.1) plus one extra equation, to wit,

$$\dot{x}_{2m} = - \sum_{\substack{1 \leq i < j < 2m \\ i \text{ odd}; j \text{ even}}} x_i x_j . \quad (3.2)$$

The proof of these formulas is by direct computation. Let us check the first formula in (3.1):

$$\begin{aligned} \{x_{2\ell-1}, z_3\}_n &= \sum_{\substack{1 \leq i < j < k \leq n \\ i, k \text{ odd}; j \text{ even}}} x_i \{x_{2\ell-1}, x_j\}_n x_k \\ &= \sum_{\substack{1 \leq i < 2\ell < k \leq n \\ i, k \text{ odd}}} x_i x_k - \sum_{\substack{1 \leq i < 2(\ell-1) < k \leq n \\ i, k \text{ odd}}} x_i x_k \\ &= \sum_{\substack{1 \leq 2\ell < k \leq n \\ k \text{ odd}}} x_{2\ell-1} x_k - \sum_{\substack{1 \leq i < 2(\ell-1) \\ i \text{ odd}}} x_i x_{2\ell-1} . \end{aligned}$$

When considering the integration of the vector field  $\mathcal{X}_{z_3}$  in dimension  $n = 2m$ , one may first integrate the vector field in dimension  $n - 1$  since (3.1) is independent of  $x_{2m}$ ; then  $x_{2m}$  can be obtained from it by simply integrating the right hand side in (3.2), because it is also independent of  $x_{2m}$ . Geometrically speaking, the vector field  $\mathcal{X}_{z_3}$  in dimension  $n = 2m - 1$  is a (Poisson) reduction of the vector field  $\mathcal{X}_{z_3}$  in dimension  $n = 2m$ . We therefore concentrate in the sequel on the case  $n = 2m - 1$ , that is, on the equations (3.1).

### 3.2 A superintegrable subsystem

A closer look at the first set of equations in (3.1) reveals that they involve only the variables  $x_k$ , with  $k$  odd. This means that the vector field  $\mathcal{X}_{z_3}$  projects to a vector field on  $\mathbb{R}^m$ , under the map

$$\begin{aligned} \phi : \quad \mathbb{R}^{2m-1} &\rightarrow \mathbb{R}^m \\ (x_1, x_2, \dots, x_{2m-1}) &\mapsto (x_1, x_3, \dots, x_{2m-1}) . \end{aligned} \quad (3.3)$$

Denoting the coordinates on  $\mathbb{R}^m$  by

$$y_i = x_{2i-1}, \quad i = 1, \dots, m, \quad (3.4)$$

the projected vector field is given by

$$\dot{y}_i = y_i \left( - \sum_{j=1}^{i-1} y_j + \sum_{j=i+1}^m y_j \right) \quad (i = 1, \dots, m) . \quad (3.5)$$

Such a vector field goes under the name of *Lotka-Volterra system* [26, 3]. In fact, it is a special Lotka-Volterra of the form  $\dot{y}_j = \sum_i c_{ij} y_i y_j$ , where the constants  $c_{ij}$  satisfy  $c_{ij} = -c_{ji}$ . This implies we have a Poisson structure and a Hamiltonian [6]. Notice that the quadratic vector field (3.1) of the  $\Upsilon$ -systems is *not* of this form because the constants  $c_{ij}$  do not satisfy the skew-symmetry property. Thus the subsystem (3.5) is a Hamiltonian vector field, but *not* with respect to the Poisson structure which is induced from the Poisson structure on  $\mathbb{R}^{2m-1}$  via the map  $\phi$  (the induced Poisson structure is trivial). Instead, consider the quadratic Poisson structure on  $\mathbb{R}^m$ , defined by

$$\{y_i, y_j\}^q := y_i y_j, \quad (3.6)$$

for any  $1 \leq i < j \leq m$ . This bracket is distinguished notationally from the bracket (2.14) by the superscript  $q$  (for quadratic) and we have omitted the dependence on the dimension. It is a well-known fact that this indeed defines a Poisson structure, see e.g. [12, Example 8.14]. Then it is clear that

$$H := z_1 = \sum_{i=1}^m y_i \quad (3.7)$$

is a Hamiltonian for the vector field (3.5). Infinitely many copies of the system (3.5) did also arise in the work of Bogoyavlenskij [3, Equation 2.8], who provided exact solutions, and, in the  $m = 4$  dimensional case, gave 3 integrals. In what follows we show, for any  $m$ , that this Hamiltonian system is superintegrable, and that it is Liouville integrable.

**Proposition 3.3.** *The Hamiltonian system (3.5) on  $\mathbb{R}^m$  admits for  $1 \leq k \leq \lfloor \frac{m+1}{2} \rfloor$  the following rational functions as constants of motion (first integrals):*

$$F_k := \begin{cases} (y_1 + y_2 + \cdots + y_{2k-1}) \frac{y_{2k+1} y_{2k+3} \cdots y_m}{y_{2k} y_{2k+2} \cdots y_{m-1}} & \text{if } m \text{ is odd,} \\ (y_1 + y_2 + \cdots + y_{2k}) \frac{y_{2k+2} y_{2k+4} \cdots y_m}{y_{2k+1} y_{2k+3} \cdots y_{m-1}} & \text{if } m \text{ is even.} \end{cases} \quad (3.8)$$

*Proof.* We assume  $m$  is odd. Then  $F_k$  can be written as

$$F_k = \sum_{\ell=1}^{2k-1} y_\ell \prod_{s=k}^{(m-1)/2} \frac{y_{2s+1}}{y_{2s}}. \quad (3.9)$$

It is easily computed from (3.5) that

$$\sum_{\ell=1}^{2k-1} \dot{y}_\ell = \left( \sum_{\ell=1}^{2k-1} y_\ell \right) \left( \sum_{\ell=2k}^m y_\ell \right) \text{ and } \frac{d}{dt} \left( \frac{y_{2s+1}}{y_{2s}} \right) = -\frac{y_{2s+1}}{y_{2s}} (y_{2s} + y_{2s+1}),$$

and so

$$\frac{d}{dt} \left( \prod_{s=k}^{(m-1)/2} \frac{y_{2s+1}}{y_{2s}} \right) = - \left( \prod_{s=k}^{(m-1)/2} \frac{y_{2s+1}}{y_{2s}} \right) \left( \sum_{\ell=2k}^m y_\ell \right).$$

The fact that  $\dot{F}_k = 0$  follows at once from these formulas. The case where  $n$  is even can be proven similarly.  $\square$

Notice that  $H = F_{\lfloor (m+1)/2 \rfloor}$ . More constants of motion can be produced by the following trick. On  $\mathbb{R}^m$  we consider the involution  $\iota$ , defined by

$$\iota(y_1, y_2, \dots, y_m) := (y_m, y_{m-1}, \dots, y_1).$$

The map  $\iota$  is anti-Poisson map, since for any  $i < j$ ,

$$\{\iota^* y_i, \iota^* y_j\}^q = \{y_{m+1-i}, y_{m+1-j}\}^q = -y_{m+1-i} y_{m+1-j} = -\iota^* \{y_i, y_j\}^q.$$

Also,  $H$  is invariant under this involution,  $\iota^* H = H$ . As a consequence,

$$(\iota^* F_k)^\cdot = \{\iota^* F_k, H\}^q = \{\iota^* F_k, \iota^* H\}^q = -\iota^* \{F_k, H\}^q = 0,$$

which proves that the rational functions  $G_k := \iota^* F_k$  ( $k = 1, \dots, \lfloor \frac{m+1}{2} \rfloor$ ) are constants of motion of (3.5). Notice that we have constructed precisely  $m - 1$  different constants of motion: when  $m$  is even, all  $F_k$  and  $G_k$  are different, except for  $F_{m/2} = H = G_{m/2}$ ; when  $m$  is odd, all  $F_k$  and  $G_k$  are different, except for  $F_{(m+1)/2} = H = G_{(m+1)/2}$  and  $F_1 = G_1$ . We note that for the 4-dimensional Lotka-Volterra system (3.5) the integrals coincide with the ones given by Bogoyavlenskij [3, (7.9)].

**Proposition 3.4.** *Let  $r := \lfloor \frac{m+1}{2} \rfloor$ , so that  $m = 2r$  when  $m$  is even and  $m = 2r - 1$  when  $m$  is odd. For  $i$  and  $j$ , satisfying  $1 \leq i, j < r$ , we have*

$$\{F_i, F_j\}^q = \{F_i, H\}^q = 0, \quad (3.10)$$

$$\{G_i, G_j\}^q = \{G_i, H\}^q = 0. \quad (3.11)$$

Moreover, the following  $m - 1$  functions are independent:

$$F_1, \dots, F_{r-1}, G_1, \dots, G_{r-1}, G_r = F_r = H, \text{ when } m \text{ is even,} \quad (3.12)$$

$$F_1 = G_1, F_2, \dots, F_{r-1}, G_2, \dots, G_r = F_r = H, \text{ when } m \text{ is odd.} \quad (3.13)$$

As a consequence,

- (1) The vector field (3.5) is superintegrable;
- (2)  $(\mathbb{R}^m, \{\cdot, \cdot\}^q, (F_1, \dots, F_{r-1}, H))$  is a Liouville integrable system;
- (3)  $(\mathbb{R}^m, \{\cdot, \cdot\}^q, (G_1, \dots, G_{r-1}, H))$  is a Liouville integrable system.

*Proof.* In Appendix A we prove that both sets of functions (3.12) and (3.13) are functionally independent. Here we prove that  $\{F_i, F_j\}^q = 0$  for all  $i$  and  $j$  satisfying  $1 \leq i, j < r$ . We do this by induction on  $m$ . For  $m = 2, 3, 4$  there is nothing to prove. Assume therefore that the formula is correct for some  $m \geq 4$ ; we show that it holds for  $m + 2$ . We denote the functions  $F_k$  which are constructed in dimension  $m$  by  $F_k^{(m)}$  and we set, as before,  $r := \lfloor (m + 1)/2 \rfloor$ . For  $k = 1, \dots, r$ , we have

$$F_k^{(m+2)} = F_k^{(m)} \frac{y_{m+2}}{y_{m+1}}. \quad (3.14)$$

Let  $1 \leq i, j < r + 1$ . In view of the above formulas and the induction hypothesis,

$$\begin{aligned} \left\{ F_i^{(m+2)}, F_j^{(m+2)} \right\}^q &= \left\{ F_i^{(m)} \frac{y_{m+2}}{y_{m+1}}, F_j^{(m)} \frac{y_{m+2}}{y_{m+1}} \right\}^q \\ &= \frac{y_{m+2}}{y_{m+1}} \left( F_i^{(m)} \left\{ \frac{y_{m+2}}{y_{m+1}}, F_j^{(m)} \right\}^q - F_j^{(m)} \left\{ \frac{y_{m+2}}{y_{m+1}}, F_i^{(m)} \right\}^q \right). \end{aligned}$$

The latter two brackets are zero for the following general reason: if  $F$  is a function which depends only on  $y_1, \dots, y_m$ , then  $\{y_{m+2}/y_{m+1}, F\} = 0$ . To show this fact, let  $1 \leq i \leq m$ . Then

$$\left\{ \frac{y_{m+2}}{y_{m+1}}, y_i \right\}^q = -\frac{y_{m+2} y_i}{y_{m+1}} + \frac{y_{m+2} y_{m+1} y_i}{y_{m+1}^2} = 0.$$

This shows that  $\left\{ F_i^{(m+2)}, F_j^{(m+2)} \right\}^q = 0$  for  $1 \leq i, j < r + 1$ , which proves (3.10), since we know from Proposition 3.3 that the functions  $F_i$  are first integrals of  $\mathcal{X}_H$ , hence are in

involution with  $H$ . Also, since  $\iota$  is an anti-Poisson map, (3.11) follows immediately from (3.10).

This shows that in both cases ( $m$  even or odd), the Hamiltonian vector field  $\mathcal{X}_H$ , which is explicitly given by (3.5), has  $m-1$  independent constants of motion, hence is superintegrable. When  $m$  is even, the rank of the Poisson structure  $\{\cdot, \cdot\}^q$  is equal to  $m$ , so that the  $r$  functions  $F_1, \dots, F_{r-1}, H$ , which are independent and in involution, define a Liouville integrable system on  $(\mathbb{R}^m, \{\cdot, \cdot\}^q)$ . When  $m$  is odd,  $F_1 = G_1$  is a Casimir function of  $\{\cdot, \cdot\}^q$  and one needs for Liouville integrability, besides the Casimir function  $(m-1)/2 = r-1$  independent functions in involution, so again the functions  $F_1, \dots, F_{r-1}, H$  define a Liouville integrable system on  $(\mathbb{R}^m, \{\cdot, \cdot\}^q)$ . This shows property (2). Property (3) is an immediate consequence of it upon using the involution  $\iota$ ; in particular, the Liouville integrable systems in (2) and (3) are isomorphic.  $\square$

It is a classical result, due to Poisson, that the Poisson bracket of two constants of motion is again a constant of motion. We give in the following proposition explicit formulas for the Poisson bracket of the constants of motion  $F_i$  and  $G_j$  of  $\mathcal{X}_H$ , as proven in Appendix B.

**Proposition 3.5.** *Let  $r := \lceil \frac{m+1}{2} \rceil$ , as in Proposition 3.4. For  $i$  and  $j$ , satisfying  $1 \leq i, j < r$ , set  $\kappa := i + j - r - 1$ . Then*

$$\{F_i, G_j\}^q = \begin{cases} -F_i G_j & \kappa < 0, \text{ } m \text{ even,} \\ 0 & \kappa < 0, \text{ } m \text{ odd,} \\ -F_{r-j} G_{r-i} & \kappa \geq 0, \text{ } m \text{ even,} \\ (-1)^\kappa \frac{\prod_{\ell=0}^{\kappa} (F_1 G_r - F_{i-\kappa+\ell} G_{j-\ell})}{\prod_{\ell=1}^{\kappa} (F_1 G_r + F_{i-\kappa+\ell-1} G_{j-\ell})} & \kappa \geq 0, \text{ } m \text{ odd.} \end{cases}$$

Since the vector field (3.5) is superintegrable, it can be quite explicitly integrated. This can be done as follows. For  $j = 1, \dots, m$ , define  $u_j := y_1 + y_2 + \dots + y_j$ . Then  $H = u_m$ . It is easy to see that in terms of the functions  $u_i$ , the Poisson structure is given by

$$\{u_i, u_j\}^q = u_i(u_j - u_i), \quad i < j,$$

in particular the Hamiltonian vector field (3.5) is in terms of these coordinates given by

$$\dot{u}_i = \{u_i, H\}^q = u_i(H - u_i). \quad (3.15)$$

Thus, the variables  $u_i$  provide a separation of variables and each one of the variables satisfies the same differential equation (3.15); its explicit integration is immediate, and can be found in [3].

### 3.3 Superintegrability and non-commutative integrability of the quadratic vector fields

We now show how the superintegrability of the  $m$ -dimensional Lotka-Volterra system, studied in Section 3.3, extend to the superintegrability (resp. non-commutative integrability) of the quadratic vector field of the  $\Upsilon$ -system in dimension  $n = 2m - 1$  (resp.  $n = 2m$ ).

**Proposition 3.6.** *When  $n$  is odd, the quadratic vector field of the  $n$ -dimensional  $\Upsilon$ -system is superintegrable.*

*Proof.* Write  $n = 2m - 1$ . Suppose first that  $m$  is odd, set  $m = 2r - 1$  and consider the following  $n - 1$  functions:

$$z_1, z_3, \dots, z_n, F_1, F_2, \dots, F_{r-1}, G_2, G_3, \dots, G_{r-1} .$$

Since the quadratic vector field is the Hamiltonian vector field  $\mathcal{X}_{z_3}$ , Theorem 2.5 implies that the functions  $z_1, z_3, z_5, \dots, z_n$  are in involution with  $z_3$ , hence are constants of motion of  $\mathcal{X}_{z_3}$ . Furthermore, according to Proposition 3.4, the above functions  $F_i$  and  $G_j$ , with (3.4), are constants of motion of  $\{\cdot, H\}^q$ , which is the projected vector field of  $\mathcal{X}_{z_3}$ , hence they are, viewed as functions on  $\mathbb{R}^n$ , constants of motion of  $\mathcal{X}_{z_3}$ . It can be shown that these  $n - 1$  functions are functionally independent (a proof is given in Appendix A). This shows that the vector field  $\mathcal{X}_{z_3}$  is superintegrable when  $m$  is odd. The proof in case  $m$  is even,  $m = 2r$ , is the same; in this case one uses the following  $n - 1$  functions:

$$z_1, z_3, \dots, z_n, F_1, F_2, \dots, F_{r-1}, G_1, G_2, \dots, G_{r-1} .$$

□

**Proposition 3.7.** *When  $n$  is even, the quadratic vector field of the  $n$ -dimensional  $\Upsilon$ -system is a non-commutative integrable system of rank 2 on  $(\mathbb{R}^n, \{\cdot, \cdot\}_n)$ .*

*Proof.* Write  $n = 2m$  and suppose first that  $m$  is odd,  $m = 2r - 1$ . Consider the following  $n - 2$  functions:

$$z_1, z_3, \dots, z_{n-1}, F_1, F_2, \dots, F_{r-1}, G_2, G_3, \dots, G_{r-1} .$$

As in the proof of Proposition 3.6,  $z_3$  is in involution with each one of these functions. Since  $z_1 = G_r = F_r = H$ , we have that  $z_1$  is also in involution with all these functions. Since these  $n - 2$  functions are independent, as is shown in Appendix A, and since the Hamiltonian vector fields of  $z_1$  and  $z_3$  are independent (at a generic point of  $\mathbb{R}^n$ ), this shows that these  $n - 2$  functions define a non-commutative integrable system of rank two. The proof in case  $m$  is even,  $m = 2r$ , is the same; one uses in this case the following  $n - 2$  functions:

$$z_1, z_3, \dots, z_{n-1}, F_1, F_2, \dots, F_{r-1}, G_1, G_2, \dots, G_{r-1} .$$

□

### 3.4 Kahan discretization

Kahan discretization was introduced as an unconventional discretisation method in [9, 10]. It seems to have quite remarkable properties in the sense of preserving geometric structures [16, 5]. In this section we consider the Kahan discretization of the quadratic Hamiltonian system (3.5). The Kahan discretization of (3.5) with step size  $2\epsilon$  is given by

$$\begin{aligned} \tilde{y}_1 - y_1 &= \epsilon y_1 (\tilde{y}_2 + \tilde{y}_3 + \dots + \tilde{y}_m) + \epsilon \tilde{y}_1 (y_2 + y_3 + \dots + y_m) , \\ \tilde{y}_2 - y_2 &= \epsilon y_2 (-\tilde{y}_1 + \tilde{y}_3 + \dots + \tilde{y}_m) + \epsilon \tilde{y}_2 (-y_1 + y_3 + \dots + y_m) , \\ &\vdots \\ \tilde{y}_m - y_m &= \epsilon y_m (-\tilde{y}_1 - \tilde{y}_2 - \dots - \tilde{y}_{m-1}) + \epsilon \tilde{y}_m (-y_1 - y_2 - \dots - y_{m-1}) . \end{aligned}$$

In what follows we introduce the variables

$$u_i := y_1 + y_2 + \dots + y_i, \quad \tilde{u}_i := \tilde{y}_1 + \tilde{y}_2 + \dots + \tilde{y}_i. \quad (3.16)$$



Thus,  $u_m = H$ , cf. (3.7), and it follows at once from summing up the above equations that  $\tilde{u}_m = u_m$ , i.e., that  $H$  is an integral of the discrete map

$$(y_1, y_2, \dots, y_m) \mapsto (\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_m). \quad (3.17)$$

We will prove that, more generally, all the integrals given by (3.8) are also integrals for this discrete system. To do this, we use the following lemma.

**Lemma 3.8.** *The Kahan discretisation of (3.5) is explicitly given by, for  $1 \leq j \leq m$ ,*

$$\tilde{y}_j = y_j \frac{(1 - \epsilon H)(1 + \epsilon H)}{(1 - \epsilon H + 2\epsilon u_{j-1})(1 - \epsilon H + 2\epsilon u_j)}, \quad (3.18)$$

where  $u_j$ , and  $H$ , are given by (3.16), and (3.7), respectively.

*Proof.* Summing up the first  $j$  equations of the above Kahan discretization, we get, upon using  $H = u_m = \tilde{u}_m$ ,

$$\tilde{u}_j - u_j = \epsilon u_j (H - \tilde{u}_j) + \epsilon \tilde{u}_j (H - u_j),$$

which can be solved linearly as

$$\tilde{u}_j = u_j \frac{1 + \epsilon H}{1 - \epsilon H + 2\epsilon u_j}. \quad (3.19)$$

The formula (3.18) follows at once from it, since  $y_j = u_j - u_{j-1}$  and  $\tilde{y}_j = \tilde{u}_j - \tilde{u}_{j-1}$ .  $\square$

We remark that equation (3.19) is the Kahan discretization of equation (3.15).

**Proposition 3.9.** *The rational functions  $F_k$ , defined in proposition 3.3, and the functions  $G_k := \iota^* F_k$  are first integrals of the Kahan discretization 3.18.*

*Proof.* We give the proof for  $m$  odd; it is essentially the same for  $m$  even. First, observe from Lemma 3.8 that

$$\frac{\tilde{y}_{2s+1}}{\tilde{y}_{2s}} = \frac{y_{2s+1}}{y_{2s}} \frac{1 - \epsilon H + 2\epsilon u_{2s-1}}{1 - \epsilon H + 2\epsilon u_{2s+1}}. \quad (3.20)$$

Therefore, using the formula (3.9) for  $F_k$ , (3.19) and (3.20) we get that

$$\begin{aligned} \tilde{F}_k &= \tilde{u}_{2k-1} \prod_{s=k}^{(m-1)/2} \frac{\tilde{y}_{2s+1}}{\tilde{y}_{2s}} \\ &= u_{2k-1} \frac{1 + \epsilon H}{1 - \epsilon H + 2\epsilon u_{2k-1}} \prod_{s=k}^{(m-1)/2} \left( \frac{y_{2s+1}}{y_{2s}} \frac{1 - \epsilon H + 2\epsilon u_{2s-1}}{1 - \epsilon H + 2\epsilon u_{2s+1}} \right) \\ &= u_{2k-1} \frac{1 + \epsilon H}{1 - \epsilon H + 2\epsilon u_m} \prod_{s=k}^{(m-1)/2} \frac{y_{2s+1}}{y_{2s}} \\ &= u_{2k-1} \prod_{s=k}^{(m-1)/2} \frac{y_{2s+1}}{y_{2s}} = F_k, \end{aligned}$$

where the last line was obtained by using that  $H = u_m$ . This shows that each  $F_k$  is a first integral. Since the discrete system is invariant under the map  $\iota$  (upon replacing  $\epsilon$  by  $-\epsilon$ ), each  $G_k$  is also a first integral.  $\square$

**Proposition 3.10.** *The Kahan discretization of (3.5) is a Poisson map.*

*Proof.* Notice that in terms of the coordinates  $u_i$ , the Poisson bracket is given by  $\{u_i, u_j\}^q = u_i(u_j - u_i)$ , for  $i < j$ ; in particular,  $\{u_i, H\}^q = u_i(H - u_i)$ . It therefore suffices to show that  $\{\tilde{u}_i, \tilde{u}_j\}^q = \tilde{u}_i(\tilde{u}_j - \tilde{u}_i)$  for  $i < j$ . Notice that  $\tilde{u}_i$  depends only on  $u_i$  and  $H$ , since

$$\tilde{u}_i = u_i \frac{1 + \epsilon H}{1 - \epsilon H + 2\epsilon u_i}. \quad (3.21)$$

We have

$$\frac{\partial \tilde{u}_i}{\partial u_i} = \frac{1 - \epsilon^2 H^2}{(1 - \epsilon H + 2\epsilon u_i)^2}, \quad \frac{\partial \tilde{u}_i}{\partial H} = \frac{2\epsilon u_i(1 + \epsilon u_i)}{(1 - \epsilon H + 2\epsilon u_i)^2}.$$

It follows that

$$\begin{aligned} \{\tilde{u}_i, \tilde{u}_j\}^q &= \frac{\partial \tilde{u}_i}{\partial u_i} \frac{\partial \tilde{u}_j}{\partial u_j} \{u_i, u_j\}^q + \frac{\partial \tilde{u}_i}{\partial u_i} \frac{\partial \tilde{u}_j}{\partial H} \{u_i, H\}^q + \frac{\partial \tilde{u}_i}{\partial H} \frac{\partial \tilde{u}_j}{\partial u_j} \{H, u_j\} \\ &= \frac{(1 - \epsilon^2 H^2) u_i (u_j - u_i)}{(1 - \epsilon H + 2\epsilon u_i)^2 (1 - \epsilon H + 2\epsilon u_j)^2} (1 - \epsilon^2 H^2 + 2\epsilon(1 + \epsilon H) u_j) \\ &= \frac{(1 + \epsilon H)^2 (1 - \epsilon H) u_i (u_j - u_i)}{(1 - \epsilon H + 2\epsilon u_i)^2 (1 - \epsilon H + 2\epsilon u_j)} \\ &= \tilde{u}_i (\tilde{u}_j - \tilde{u}_i). \end{aligned}$$

as was to be shown.  $\square$

For a discrete map in dimension  $n$  the existence of  $n - 1$  integrals is not enough to claim (super)integrability.

**Definition 3.11.** An  $n$ -dimensional map is *superintegrable* if it has  $n - 1$  constants of motion and it is measure preserving. A  $n = 2r + s$ -dimensional map on a Poisson manifold, which respects the Poisson structure of rank  $2r$  is *Liouville integrable* if there are  $n - r = r + s$  functionally independent constants of motion in involution, cf. [4, 13, 25].

It is known that for a symplectic map  $L : y \mapsto \tilde{y}$  with structure matrix  $\Omega$  we have

$$dL.\Omega.dL^T = \tilde{\Omega},$$

where  $dL$  is the Jacobian matrix of  $L$ . It yields  $(\det(dL))^2 = \det(\tilde{\Omega})/\det(\Omega)$ . By calculating the determinant of our structure matrix when  $m$  is even, we obtain the following result. The result also holds for odd  $m$ , which is why we provide a direct proof, i.e. without assuming symplecticity.

**Proposition 3.12.** *The Kahan discretization (3.18) is measure preserving, with measure*

$$\frac{1}{y_1 y_2 \dots y_m} dy_1 \wedge dy_2 \wedge \dots \wedge dy_m.$$

*Proof.* It is easy to see that

$$\frac{\partial(\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_m)}{\partial(y_1, y_2, \dots, y_m)} = \frac{\partial(\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_m)}{\partial(u_1, u_2, \dots, u_m)} \cdot \frac{\partial(u_1, u_2, \dots, u_m)}{\partial(y_1, y_2, \dots, y_m)}. \quad (3.22)$$

Since  $\tilde{y}_i = \tilde{u}_i - \tilde{u}_{i-1}$ , we have

$$\frac{\partial(\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_m)}{\partial(u_1, u_2, \dots, u_m)} = \frac{\partial(\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_m)}{\partial(u_1, u_2, \dots, u_m)} - \frac{\partial(0, \tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_{m-1})}{\partial(u_1, u_2, \dots, u_m)} := A$$

Entries of  $A$  are obtained by direct calculation and given as follow

$$A[i, j] = \begin{cases} \frac{\partial \tilde{u}_i}{\partial u_i} = \frac{1 - \epsilon^2 u_m^2}{(1 - \epsilon u_m + 2\epsilon u_i)^2} & \text{if } j = i, \\ -\frac{\partial \tilde{u}_{i-1}}{\partial u_{i-1}} = -\frac{1 - \epsilon^2 u_m^2}{(1 - \epsilon u_m + 2\epsilon u_{i-1})^2} & \text{if } j = i - 1, \\ \frac{\partial \tilde{u}_i}{\partial u_m} - \frac{\partial \tilde{u}_{i-1}}{\partial u_m} = \frac{2\epsilon u_i(1 + \epsilon u_i)}{(1 - \epsilon u_m + 2\epsilon u_i)^2} - \frac{2\epsilon u_{i-1}(1 + \epsilon u_{i-1})}{(1 - \epsilon u_m + 2\epsilon u_{i-1})^2} & \text{if } j = m, \\ 0 & \text{otherwise.} \end{cases} \quad (3.23)$$

To calculate the determinant of  $A$ , we divide the  $j^{\text{th}}$  column by  $\frac{1 - \epsilon^2 u_m^2}{(1 - \epsilon u_m + 2\epsilon u_i)}$  for all  $j < m$  and then adding the first  $i_1$  rows to the  $i^{\text{th}}$  row. We obtain an upper triangular matrix with 1 on the diagonal. Therefore, one gets

$$\det(A) = \frac{(1 - \epsilon^2 u_m^2)^{m-1}}{(1 - \epsilon u_m + 2\epsilon u_1)^2 (1 - \epsilon u_m + 2\epsilon u_2)^2 \dots (1 - \epsilon u_m + 2\epsilon u_{m-1})^2} \quad (3.24)$$

$$= \frac{\tilde{y}_1 \tilde{y}_2 \dots \tilde{y}_m}{y_1 y_2 \dots y_m} \quad (3.25)$$

On the other hand, the determinant of  $\frac{\partial(u_1, u_2, \dots, u_m)}{\partial(y_1, y_2, \dots, y_m)}$  in (3.22) is 1. Thus,

$$\frac{d\tilde{y}_1 \wedge \dots \wedge d\tilde{y}_m}{\tilde{y}_1 \dots \tilde{y}_m} = \frac{1}{\tilde{y}_1 \dots \tilde{y}_m} \frac{\partial(\tilde{y}_1, \dots, \tilde{y}_m)}{\partial(y_1, \dots, y_m)} dy_1 \wedge \dots \wedge dy_m = \frac{dy_1 \wedge \dots \wedge dy_m}{y_1 \dots y_m},$$

which shows the Kahan discretization (3.18) is measure preserving, with the given measure.  $\square$

As a direct consequence of Propositions 3.4, 3.9, 3.10, and 3.12 we get the following result.

**Proposition 3.13.** *The Kahan discretization (3.18) is both superintegrable, and Liouville integrable.*

Finally we'd like to remark that due to the preservation of both the Poisson structure, and the Hamiltonian, the Kahan discretization (3.18) is the time advance map for the exact Hamiltonian system (3.5) up to a reparametrization of time [29].

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## A Functional independence of the integrals of the quadratic vector fields

We first show that when  $m = 2r$  is even, the  $m - 1$  functions  $F_1, \dots, F_{r-1}, G_1, \dots, G_{r-1}, H = F_r = G_r$  are (functionally) independent, as was asserted in the proof of Proposition 3.4. Let us write  $\mathbf{1}$  as a shorthand for the point  $(1, 1, \dots, 1) \in \mathbb{R}^{2r}$ . At this point, we have that

$$\frac{\partial F_k}{\partial y_i}(\mathbf{1}) = \frac{\partial G_k}{\partial y_{2r+1-i}}(\mathbf{1}) = \begin{cases} 1 & \text{if } 1 \leq i \leq 2k, \\ 2k & \text{if } 2k < i \leq 2r \text{ and } i \text{ is even,} \\ -2k & \text{if } 2k < i \leq 2r \text{ and } i \text{ is odd.} \end{cases} \quad (\text{A.1})$$

Define rational functions  $K_1, \dots, K_{2r-1}$  by

$$K_j := \begin{cases} F_{(j-1)/2} - H & \text{if } j \text{ is odd,} \\ G_{r+1-j/2} - (r+1-j/2)G_1 & \text{if } j \text{ is even,} \end{cases}$$

where  $F_0 := 0$ , so that  $K_1 = -H$ . We show that the Jacobian of these functions is of maximal rank  $(2r - 1)$  at  $\mathbf{1}$ . On the one hand, (A.1) implies that

$$\frac{\partial K_j}{\partial y_i}(\mathbf{1}) = 0 \text{ for } \begin{cases} j \text{ odd, } i < j, \\ j \text{ even, } i < j - 1. \end{cases}$$

On the other hand, for  $i = 1, \dots, r - 1$ , the matrices

$$\begin{pmatrix} \frac{\partial K_{2i-1}}{\partial y_{2i-1}}(\mathbf{1}) & \frac{\partial K_{2i-1}}{\partial y_{2i}}(\mathbf{1}) \\ \frac{\partial K_{2i}}{\partial y_{2i-1}}(\mathbf{1}) & \frac{\partial K_{2i}}{\partial y_{2i}}(\mathbf{1}) \end{pmatrix} = \begin{pmatrix} 1 - 2i & 2i - 3 \\ 2i - 1 - 2r & 2r - 2i + 3 \end{pmatrix}$$

are all non-singular and  $\frac{\partial K_{2r-1}}{\partial y_{2r-1}}(\mathbf{1}) = -(2r - 1) \neq 0$ . It follows that the Jacobian of the functions  $K_1, \dots, K_{2r-1}$ , and hence of the functions  $F_1, \dots, F_{r-1}, G_1, \dots, G_{r-1}, H = F_r = G_r$ , is of maximal rank at the point  $\mathbf{1}$ . This shows that the latter functions are functionally independent on  $\mathbb{R}^{2r}$ . The proof that when  $m = 2r - 1$  is odd, the  $m - 1$  functions  $F_1 = G_1, F_2, \dots, F_{r-1}, G_2, \dots, G_r = F_r = H$  are functionally independent goes along the same lines.

As was stated in Section 33.3, the above rational functions  $K_i$  remain functionally independent when they are viewed as functions on  $\mathbb{R}^n$  and the odd polynomials  $z_3, z_5, \dots$  are added to them; here  $n = 2m - 1$  or  $n = 2m$ , depending on whether  $n$  is even (Proposition 3.6) or odd (Proposition 3.7). Since the functions  $K_i$  depend only on the variables  $y_i = x_{2i-1}$ , i.e., are independent of the variables  $x_{2i}$ , we only need to verify that the Jacobian determinant of the polynomials  $z_3, z_5, \dots$  with respect to the variables  $x_2, x_4, \dots$  is non-zero (at one point at least). Precisely, when  $n$  is odd (resp.  $n$  is even) one needs to check that the rank of the following Jacobian matrices is maximal:

$$\frac{\partial(z_3, z_5, \dots, z_n)}{\partial(x_2, x_4, \dots, x_{n-1})}, \quad \text{resp.} \quad \frac{\partial(z_3, z_5, \dots, z_{n-1})}{\partial(x_2, x_4, \dots, x_{n-2})}.$$

The proof is very similar to the proof of Proposition 2.1, which shows that the Jacobian matrix

$$\frac{\partial(z_1, z_2, \dots, z_n)}{\partial(x_1, x_2, \dots, x_n)}$$

has maximal rank.

## B The brackets between functions $F$ and $G$

We outline the proof of Proposition 3.5, which proceeds by induction.

*Proof.* For  $m < 3$  there is nothing to be checked, while for  $m = 3$  it is contained in Proposition 3.4. For  $m = 4$  we only need to check that  $\{F_1, G_1\}^q = -F_1 G_1$ , which is easily done with the following explicit formulas:  $F_1 = (y_1 + y_2)y_4/y_3$  and  $G_1 = (y_3 + y_4)y_1/y_2$ . Assuming that the formulas hold for some  $m \geq 4$  one shows that they also hold for  $m + 2$ . As in the proof of Proposition 3.4, we denote the functions  $F_k$  and  $G_k$  which are constructed in dimension  $m$  by  $F_k^{(m)}$  and  $G_k^{(m)}$  and we set, as before,  $r := [(m + 1)/2]$ . We have, besides (3.14),

$$G_j^{(m+2)} = G_{j-1}^{(m)} + (y_{m+1} + y_{m+2}) \prod_{k=1}^{r-j+1} \frac{y_{2k-1}}{y_{2k}},$$

and so, in order to compute  $\{F_i^{(m+2)}, G_j^{(m+2)}\}^q$ , we only need to compute the following types of Poisson brackets:

$$\begin{aligned} & \left\{ F_i^{(m)}, G_{j-1}^{(m)} \right\}^q, \text{ which is given by the induction hypothesis;} \\ & \left\{ F_i^{(m)}, y_{m+1} + y_{m+2} \right\}^q = F_i^{(m)}(y_{m+1} + y_{m+2}), \text{ see below;} \\ & \left\{ F_i^{(m)}, \prod_{k=1}^{r-j+1} \frac{y_{2k-1}}{y_{2k}} \right\}^q, \text{ see below;} \\ & \left\{ \frac{y_{m+2}}{y_{m+1}}, G_{j-1}^{(m)} \right\}^q = 0, \text{ since } G_{j-1}^{(m)} \text{ is independent of } y_{m+1}, y_{m+2}; \\ & \left\{ \frac{y_{m+2}}{y_{m+1}}, y_{m+1} + y_{m+2} \right\}^q = -\frac{y_{m+2}}{y_{m+1}}(y_{m+1} + y_{m+2}); \\ & \left\{ \frac{y_{m+2}}{y_{m+1}}, \prod_{k=1}^{r-j+1} \frac{y_{2k-1}}{y_{2k}} \right\}^q = 0, \text{ since each } \frac{y_{2k-1}}{y_{2k}} \text{ is independent of } y_{m+1}, y_{m+2}. \end{aligned}$$

In order to show that  $\{F_i^{(m)}, y_{m+1} + y_{m+2}\}^q = F_i^{(m)}(y_{m+1} + y_{m+2})$ , it suffices to observe that  $\{y_i, y_{m+1} + y_{m+2}\}^q = y_i(y_{m+1} + y_{m+2})$  when  $i \leq m$  and that  $F_i^{(m)}$  is homogeneous of degree 1 and depends on the variables  $y_1, \dots, y_m$  only. Finally, when  $m$  is even,

$$\left\{ F_i^{(m)}, \prod_{k=1}^{r-j+1} \frac{y_{2k-1}}{y_{2k}} \right\}^q = \begin{cases} -F_i^{(m)} \prod_{k=1}^{r-j+1} \frac{y_{2k-1}}{y_{2k}} & i + j \leq r + 1, \\ -F_{r+1-j}^{(m)} \prod_{k=1}^i \frac{y_{2k-1}}{y_{2k}} & i + j > r + 1, \end{cases}$$

as follows at once from

$$\left\{ F_i^{(m)}, \frac{y_{2k-1}}{y_{2k}} \right\}^q = \begin{cases} -\frac{y_{2k-1}}{y_{2k}}(y_{2k} + y_{2k-1}) \prod_{s=i+1}^r \frac{y_{2s}}{y_{2s-1}} & k \leq i, \\ 0 & k > i; \end{cases}$$

when  $m$  is odd,

$$\left\{ F_i^{(m)}, \prod_{k=1}^{r-j+1} \frac{y_{2k-1}}{y_{2k}} \right\}^q = \begin{cases} 0 & i+j \leq r+1, \\ -\sum_{k=1}^{2i-2} \frac{F_1 y_k}{y_{2i-1}} \prod_{s=1}^{i+j-r-2} \frac{y_{2i-2s}}{y_{2i-2s-1}} & i+j > r+1. \end{cases}$$

as follows at once from

$$\left\{ F_i^{(m)}, \frac{y_{2k-1}}{y_{2k}} \right\}^q = \begin{cases} -\frac{y_{2k-1}(y_{2k} + y_{2k-1})}{y_{2k}} \prod_{s=i+1}^r \frac{y_{2s-1}}{y_{2s-2}} & k < i, \\ \frac{y_{2i-1}}{y_{2i}} \sum_{j=1}^{2i-2} y_j \prod_{s=i+1}^r \frac{y_{2s-1}}{y_{2s-2}} & k = i, \\ 0 & k > i. \end{cases}$$

Each one of these formulas is proven easily by direct computation.  $\square$

## C Lax pairs for the $\Upsilon$ -systems

We prove in this appendix that the Hamiltonian vector fields defined in Section 22.3 can be written in the Lax form  $\dot{L} = [L, B]$ , where the matrices  $L$  and  $B$  were given in Section 22.4. We do this for the systems associated to  $\mathbf{F}'$ ; for the case of  $\mathbf{F}$ , the proof is very similar. We start with the even case,  $n = 2m$ . Recall that the Lax operator  $L = L'_{2m}$  is in this case given by (2.17) and that the matrix  $B = B'_{2m,k}$  is, for even  $k$  given by (2.22). Recall also (from Section 22.4) that this matrix  $L$  is invertible and that its inverse is given by the tridiagonal matrix  $M$ , with entries  $m_{ij} := x_j^{-1}(\delta_{i,j+1} - \delta_{j,i+1})$ . When  $L$  satisfies the above Lax equation, then its inverse  $M$  satisfies the Lax equation  $\dot{M} = [M, B]$  and vice versa. It is therefore sufficient to prove the latter Lax equation. We first compute its left hand side. Since the polynomials  $\Upsilon_r^{a,b}$  depend linearly on the variables  $x_k$ , one easily computes from (2.1) that

$$\frac{\partial z_k}{\partial x_i} = \sum_{\substack{0 \leq r < k \\ r+1 \equiv i}} \Upsilon_r^{1,i-1} \Upsilon_{k-r-1}^{i+1,n}. \quad (\text{C.1})$$

Therefore, we obtain for the Hamiltonian vector field associated to  $z_k$ ,

$$\begin{aligned} \dot{x}_i &= \{x_i, z_k\} = \frac{\partial z_k}{\partial x_{i+1}} - \frac{\partial z_k}{\partial x_{i-1}} = \sum_{\substack{0 \leq r < k \\ r \equiv i}} \left( \Upsilon_r^{1,i} \Upsilon_{k-r-1}^{i+2,n} - \Upsilon_r^{1,i-2} \Upsilon_{k-r-1}^{i,n} \right); \\ \dot{m}_{ij} &= \frac{\delta_{j,i+1} - \delta_{i,j+1}}{x_j^2} \sum_{\substack{0 \leq r < k \\ r \equiv j}} \left( \Upsilon_r^{1,j} \Upsilon_{k-r-1}^{j+2,n} - \Upsilon_r^{1,j-2} \Upsilon_{k-r-1}^{j,n} \right). \end{aligned}$$

This formula is to be compared with the  $(i, j)$ -th entry of the commutator  $[M, B]$ , i.e., we need to show that

$$\dot{m}_{ij} = \frac{b_{i-1,j}}{x_{i-1}} - \frac{b_{i+1,j}}{x_{i+1}} - \frac{b_{i,j+1}}{x_j} + \frac{b_{i,j-1}}{x_j}. \quad (\text{C.2})$$

This is obvious when  $i > j + 1$  and when  $i = j$ , because in these cases all terms in (C.2) are zero. Let us first show that the right hand side of (C.2) is also zero when  $i < j - 1$ . For

simplicity, we assume that  $i \neq 1$  and that  $j \neq n$ . If we substitute the values of the  $b_{ij}$  and we collect on the one hand the first two terms and on the other hand the last two terms of (C.2), then we get

$$\sum_{\substack{2 \leq r \leq k-2 \\ r \equiv i+1}} \left( \Upsilon_{r-1}^{1,i} - \Upsilon_{r-1}^{1,i-2} \right) \Upsilon_{k-2-r}^{j+1,n} + \frac{x_i}{x_j} \sum_{\substack{1 \leq r \leq k-3 \\ r \equiv i}} \Upsilon_{r-1}^{1,i-1} \left( \Upsilon_{k-2-r}^{j+2,n} - \Upsilon_{k-2-r}^{j,n} \right).$$

Using the first recursion relation in (2.3) on the first term and the second recursion relation in (2.3) on the second term, we find that both terms cancel out. We next consider the case  $i = j + 1$ . For simplicity we suppose that  $j \neq 1$ . We need to prove that  $(b_{j,j} - b_{j+1,j+1})/x_j = \dot{m}_{j+1,j}$ . Written out, it means that we need to prove that

$$x_j \sum_{\substack{0 \leq r \leq k-2 \\ r \equiv j}} \Upsilon_r^{1,j-2} \Upsilon_{k-r-2}^{j+1,n} - x_j \sum_{\substack{0 \leq r \leq k-2 \\ r \equiv j+1}} \Upsilon_r^{1,j-1} \Upsilon_{k-r-2}^{j+2,n} + \sum_{\substack{0 \leq r < k \\ r \equiv j}} \left( \Upsilon_r^{1,j} \Upsilon_{k-r-1}^{j+2,n} - \Upsilon_r^{1,j-2} \Upsilon_{k-r-1}^{j,n} \right)$$

is zero. If we combine the second and third term, and we use the second recursion relation in (2.3) to simplify the sum of the first and last term, we get

$$\sum_{\substack{1 \leq r \leq k-1 \\ r \equiv j}} \left( \Upsilon_r^{1,j} - x_j \Upsilon_{r-1}^{1,j-1} \right) \Upsilon_{k-r-1}^{j+2,n} - \sum_{\substack{1 \leq r \leq k-1 \\ r \equiv j}} \Upsilon_r^{1,j-2} \Upsilon_{k-r-1}^{j+2,n}. \quad (\text{C.3})$$

In fact, when  $j$  is even, one gets two other terms, to wit  $(\Upsilon_0^{1,j} - \Upsilon_0^{1,j-2})\Upsilon_{k-1}^{j+2,n}$ , but they cancel out (recall that we supposed that  $j \neq 1$ ). The fact that (C.3) equals zero follows at once from the first recursion relation in (2.3). To finish, we consider the remaining case  $i = j - 1$ . We need to prove that  $b_{i-1,i+1}/x_{i-1} - b_{i+1,i+1}/x_{i+1} - (b_{i,i+2} - b_{i,i})/x_{i+1} = \dot{m}_{i,i+1}$ . Written out, it means that we need to show that the following sum of six terms is zero:

$$\begin{aligned} & - x_{i+1}^2 \sum_{\substack{1 \leq r \leq k-2 \\ r \equiv i+1}} \Upsilon_{r-1}^{1,i-2} \Upsilon_{k-2-r}^{i+2,n} - x_{i+1} \sum_{\substack{1 \leq r \leq k-2 \\ r \equiv i+1}} \Upsilon_r^{1,i-1} \Upsilon_{k-2-r}^{i+2,n} + x_i x_{i+1} \sum_{\substack{1 \leq r \leq k-2 \\ r \equiv i}} \Upsilon_{r-1}^{1,i-1} \Upsilon_{k-2-r}^{i+3,n} \\ & + x_{i+1} \sum_{\substack{1 \leq r \leq k-2 \\ r \equiv i}} \Upsilon_r^{1,i-2} \Upsilon_{k-2-r}^{i+1,n} - \sum_{\substack{1 \leq r < k \\ r \equiv i+1}} \Upsilon_r^{1,i+1} \Upsilon_{k-1-r}^{i+3,n} + \sum_{\substack{1 \leq r < k \\ r \equiv i+1}} \Upsilon_r^{1,i-1} \Upsilon_{k-1-r}^{i+1,n}. \end{aligned}$$

As before, we combine the terms in pairs and apply the second recursion relation in (2.3): terms 1 and 4 yield term I below, terms 2 and 6 yield II and terms 3 and 5 yield III (to obtain III one uses the recursion relation twice)<sup>3</sup>:

$$\begin{aligned} \text{I} &= x_{i+1} \sum_{\substack{1 \leq r \leq k-1 \\ r \equiv i+1}} \Upsilon_{r-1}^{1,i-2} \Upsilon_{k-1-r}^{i+3,n}, & \text{II} &= \sum_{\substack{1 \leq r \leq k-2 \\ r \equiv i+1}} \Upsilon_r^{1,i-1} \Upsilon_{k-1-r}^{i+3,n} + \Upsilon_{k-1}^{1,i-1} \delta_{i \equiv 0} \\ \text{III} &= - \sum_{\substack{2 \leq r \leq k-1 \\ r \equiv i+1}} \Upsilon_r^{1,i} \Upsilon_{k-1-r}^{i+3,n} - x_{i+1} \sum_{\substack{2 \leq r \leq k-1 \\ r \equiv i+1}} \Upsilon_{r-1}^{1,i-2} \Upsilon_{k-1-r}^{i+3,n} - \Upsilon_0^{1,i+1} \Upsilon_{k-1}^{i+3,n} \delta_{i \equiv 1} - \Upsilon_1^{1,i+1} \Upsilon_{k-2}^{i+3,n} \delta_{i \equiv 0}. \end{aligned}$$

If we add up II with the first and third term in III, only the single boundary term  $\Upsilon_1^{1,i-1} \Upsilon_{k-2}^{i+3,n} \delta_{i \equiv 0}$  remains. Similarly, adding up I and the second term in III leads to the single boundary term

<sup>3</sup>For integers  $a, b$  we use the Kronecker-like notation  $\delta_{a \equiv b}$ , which is 1 when  $a \equiv b \pmod{2}$  and 0 otherwise.

$x_{i+1}\Upsilon_0^{1,i-2}\Upsilon_{k-2}^{i+3,n}\delta_{i=0}$ . So it remains to be shown that the sum of these terms with the last term of III is zero, i.e., if  $i$  is even then

$$\left(\Upsilon_1^{1,i-1} + x_{i+1}\Upsilon_0^{1,i-2} - \Upsilon_1^{1,i+1}\right)\Upsilon_{k-2}^{i+3,n} = 0.$$

That this is so follows at once from  $\Upsilon_0^{1,i-2} = 1$  and  $\Upsilon_1^{1,i+1} = z_1^{(i+1)} = x_1 + x_3 + \dots + x_{i+1}$ . This proves the Lax equations for  $\mathbf{F}'$  when  $n$  is even.

We derive from the above result the proof for  $\mathbf{F}'$  when  $n = 2m - 1$  is odd. Recall from Section 22.4 that the matrices  $L'_{2m-1}$  and  $B'_{2m-1,2k}$  (for  $k = 1, \dots, m - 1$ ) were obtained by slightly modifying the matrices  $L'_{2m}$  and  $B'_{2m,2k}$  from the previous case. We analyse how these modifications affect on the one hand the vector field and on the other hand the commutator, which are the left and right hand sides of the Lax equation

$$\mathcal{X}_{z_{2k}}L'_{2m-1} = [L'_{2m-1}, B'_{2m-1,2k}]. \quad (\text{C.4})$$

Before doing this, notice that since by definition the last rows of  $L'_{2m-1}$  and of  $B'_{2m-1,2k}$  are zero, the last row of both sides in (C.4) is zero. Similarly, there is a single non-zero entry in the last column of  $L'_{2m-1}$ , and none in the last column of  $B'_{2m-1,2k}$ , so one only needs to check that  $\{x_{2m-1}, z_{2k}\}_{2m-1} = -z_{2k}^{(2m-3)}$ . On the one hand,  $\{x_{2m-1}, z_{2k}\}_{2m-1} = -\partial z_{2k}^{(2m-1)}/\partial x_{2m-2}$ ; on the other hand, it follows from the recursion relation (2.6) that  $z_{2k}^{(2m-1)} = z_{2k}^{(2m-3)} + x_{2m-2}z_{2k-1}^{(2m-3)}$ . Next, notice that the remaining entries of the second-to-last rows and columns of both sides of (C.4) are zero, since they are already zero in  $L'_{2m}$  and are by definition zero in  $B'_{2m-1,2k}$  (except for the diagonal entry). For the other entries  $(i, j)$  of the matrices we compare the vector fields. The recursion relation (2.6) yields  $z_{2k}^{(2m)} = z_{2k}^{(2m-1)} + x_{2m}z_{2k-1}^{(2m-1)}$ , which implies that

$$\left\{x_i, z_{2k}^{(2m-1)}\right\}_{2m-1} = \left(\left\{x_i, z_{2k}^{(2m)}\right\}_{2m}\right)_{|x_{2m}=0},$$

since  $i < 2m - 1$ . This means that, except for its two last rows and columns, the matrix  $\mathcal{X}_{z_{2k}}L'_{2m-1}$  is obtained from  $\mathcal{X}_{z_{2k}}L'_{2m}$  by substituting 0 for  $x_{2m}$ . We need to check that this is also so for the corresponding submatrix of  $[L'_{2m-1}, B'_{2m-1,2k}]$ . Let  $1 \leq i, j \leq 2m - 2$ . Then  $(L'_{2m-1})_{i,\ell}(B'_{2m-1,2k})_{\ell,j} = ((L'_{2m})_{i,\ell}(B'_{2m,2k})_{\ell,j})_{|x_{2m}=0}$  and  $(B'_{2m-1,2k})_{i,\ell}(L'_{2m-1})_{\ell,j} = (B'_{2m,2k})_{i,\ell}((L'_{2m})_{\ell,j})_{|x_{2m}=0}$  for  $\ell = 1, \dots, 2m$ : for  $\ell = 1, \dots, 2m - 2$  this is true by definition, while for  $\ell = 2m - 1$  and  $\ell = m$  both sides are zero.

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