SINGULAR FIBER OF THE MUMFORD SYSTEM AND RATIONAL SOLUTIONS TO THE KDV HIERARCHY

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Abstract. We study the singular iso-level manifold $M_g(0)$ of the genus g Mumford system associated to the spectral curve $y^2 = x^{2g+1}$. We show that $M_g(0)$ is stratified by $g+1$ open subvarieties of additive algebraic groups of dimension $0, 1, \ldots, g$ and we give an explicit description of $M_g(0)$ in terms of the compactification of the generalized Jacobian. As a consequence, we obtain an effective algorithm to compute rational solutions to the genus g Mumford system, which is closely related to rational solutions of the KdV hierarchy.

1. Introduction

The notion of algebraic integrability has been introduced by Adler and van Moerbeke in order to provide a natural context in which basically all classical examples of integrable systems naturally fit (after complexification) and they have developed techniques for studying the geometry and the explicit integration of these systems [2, 3, 4]. The main feature of an algebraic completely integrable system (a.c.i. system) is that the *generic* fiber of its complex momentum map (the map which is defined by the Poisson commuting integrals) is an affine part of an Abelian variety (compact complex algebraic torus); in addition, the corresponding Hamiltonian vector fields are demanded to define translation invariant vector fields on these tori. One important consequence is that the integration of the equations of motion, starting from a *generic* point, can be done in terms of theta functions, such as the classical Riemann theta function. A widely known example of an a.c.i. system is the Euler top, which Euler integrated in terms of elliptic functions.

Particular *special (non-generic)* fibers of a the moment map of an a.c.i. system are in general not affine parts of an Abelian variety. According to a conjecture, stated in [4, p. 155], such a fiber is made up by affine parts of one or several algebraic groups, defined by the flows of the integrable vector fields. The solutions starting from a point on such a fiber are then expressed in terms of a degeneration of the theta function, such as exponential or rational functions. When the generic fiber of the a.c.i. system is the Jacobian of a Riemann surface, so that the solution is expressed in terms of its Riemann theta function, one is tempted to relate the algebraic groups that make up a special fiber to a generalized

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Jacobian, i.e., the Jacobian of a singular algebraic curve. Then the function theory of these Jacobians provides the algebraic functions in which the corresponding special solution can be expressed. In the case which we will study in this paper, the zero-fiber of the genus g Mumford system, the singular curve is of the form $y^2 = x^{2n+1}$ (where $n \leqslant g$) and the entire zero-fiber admits, according to a result by Beauville [8], a natural description as an affine part of the compactification of the generalized Jacobian of the curve $y^2 = x^{2g+1}$. We will show that the corresponding solutions of the Mumford system are rational functions of all time variables and we will give explicit formulas for these solutions. See [7, 11, 12] for other works on integrable systems involving generalized Jacobians.

Recall [17, 22] that for a fixed positive integer g, the phase space M_g of the Mumford system is given by

$$M_g = \left\{ \ell(x) = \begin{pmatrix} v(x) & w(x) \\ u(x) & -v(x) \end{pmatrix} \;\middle|\; \begin{array}{l} u(x) = x^g + u_{g-1}x^{g-1} + \cdots + u_0, \\ v(x) = v_{g-1}x^{g-1} + \cdots + v_0, \\ w(x) = x^{g+1} + w_g x^g + \cdots + w_0 \end{array} \right\} (\cong \mathbf{C}^{3g+1}), \tag{1.1}$$

equipped with a Poisson structure $\{\cdot\,,\cdot\}$. We have the momentum map

$$\Phi_g : M_g \to H_g \;:\; \ell(x) \mapsto -\det(\ell(x)),$$

where $H_g \cong \mathbf{C}^{2g+1}$ is given by

$$H_g = \{h(x) = x^{2g+1} + h_{2g}x^{2g} + h_{2g-1}x^{2g-1} + \cdots + h_0 \mid h_0, \ldots, h_{2g} \in \mathbf{C}\}. \tag{1.2}$$

Out of the $2g+1$ independent functions $h_0, \ldots, h_{2g+1}$ on M_g, $g+1$ functions $h_g, \ldots, h_{2g}$ are Casimirs, and the g other functions $h_0, \ldots, h_{g-1}$ define commuting Hamiltonian vector fields $\mathcal{X}_1, \cdots, \mathcal{X}_g$. This implies, since the generic rank of $\{\cdot\,,\cdot\}$ is $2g$ on M_g, that the system $(M_g, \{\cdot\,,\cdot\}, \Phi_g)$ is a Liouville integrable system. For $h(x) \in H_g$, let $C_g(h)$ denote the integral projective (possibly singular) hyperelliptic curve of (arithmetic) genus g, given by the completion of the affine curve $y^2 = h(x)$ with one smooth point at infinity. The main feature of the Mumford system is that, when $C_g(h)$ is non-singular, there is an isomorphism between the level set $M_g(h) := \Phi_g^{-1}(h)$ and the complement of the theta divisor in the Jacobian variety $J_g(h)$ of $C_g(h)$, which transforms the Hamiltonian vector fields $\mathcal{X}_1, \cdots, \mathcal{X}_g$ into the translation invariant vector fields on $J_g(h)$. This shows that the Mumford system is a.c.i. For singular curves, according to Beauville [8], the same result holds true, upon replacing the Jacobian by the compactified generalized Jacobian (and the theta divisor by its completion in the latter).

In this paper we give a precise and explicit description of the zero-fiber of the Mumford system, which is the fiber of Φ_g over the very special point $h(x) = x^{2g+1}$ in H_g, for which the spectral curve $C_g := C_g(x^{2g+1})$ becomes a singular curve given by $y^2 = x^{2g+1}$. Our results can be summarized as follows. (See Theorems 4.2, 4.4 and Proposition 5.7 for (1)-(3).)

(1) The level set $M_g(x^{2g+1})$ is stratified by $g+1$ smooth affine varieties, which are invariant for the flows of the vector fields $\mathcal{X}_1, \ldots, \mathcal{X}_g$; they are of dimension $k = 0, 1, \ldots, g$.
(2) Let $k \in \{0, 1, \ldots, g\}$. There is an isomorphism between the (unique) k-dimensional invariant manifold in $M_g(x^{2g+1})$ and the complement of the 'theta divisor' Θ_k in the generalized Jacobian J_k of C_k, which linearizes the vector fields $\mathcal{X}_1, \ldots, \mathcal{X}_k$. (The vector fields $\mathcal{X}_{k+1}, \ldots, \mathcal{X}_g$ vanish.) On the other hand, we construct explicitly an isomorphism between J_k and the additive group $\mathbf{C}^k$, by which Θ_k is transformed to the zero locus of an (explicitly constructed) polynomial function τ_k on $\mathbf{C}^k$. Combined, for $k = g$, this yields a rational solution to the Mumford system in terms of τ_g and its derivatives.
(3) The entire level set $M_g(x^{2g+1})$ is isomorphic to the complement of the 'completed theta divisor' $\bar{\Theta}_g$ in the compactification $\bar{J}_g$ of J_g. The vector fields $\mathcal{X}_1, \ldots, \mathcal{X}_g$ are transformed to the ones induced by the natural action of J_g on $\bar{J}_g$ via this isomorphism.

The rational solutions, obtained in (2), turn out to be exactly same as the rational solutions to the Korteweg-de Vries (KdV) hierarchy constructed in [1, 5, 6, 14, 18, 20]. This is not surprising, since Mumford's original motivation for constructing the Mumford system is the fact that *every* solution to the Mumford system yields a solution to the KdV hierarchy [17, p. 3.203]. We therefore recover the rational solutions of the KdV hierarchy by using an adapted version of the Abel-Jacobi map within the finite-dimensional framework of the Mumford system.

Outline of the paper. In §2, we briefly review the basic facts about the Mumford system. §3 is devoted to a detailed analysis of the generalized Jacobian J_g of C_g and its compactification $\bar{J}_g$. We then apply in §4 the results of §3 to the Mumford system. In §5, we give an algorithm to produce rational solutions for the Mumford system. In §6, we study the relation to the KdV hierarchy.

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2. The Mumford system

In this section, we recall the basic facts about the Mumford system ([17], [22, Ch. VI.4]). Throughout the section, g is a fixed positive integer.

2.1. **Hamiltonian structure and integrability.**

The phase space M_g defined in (1.1) of the Mumford system is equipped with the Poisson structure defined by (see [22, Ch. VI

(4.4)])

$$
\begin{aligned}
\{u(x), u(z)\} &= \{v(x), v(z)\} = 0,\\
\{u(x), v(z)\} &= \frac{u(x)-u(z)}{x-z},\\
\{u(x), w(z)\} &= -2\frac{v(x)-v(z)}{x-z},\\
\{v(x), w(z)\} &= \frac{w(x)-w(z)}{x-z} - u(x),\\
\{w(x), w(z)\} &= 2\big(v(x)-v(z)\big).
\end{aligned}
$$

The natural coordinates $h_0, \dots, h_{2g}$ on H_g (1.2) can be regarded as polynomial functions on M_g. These functions are pairwise in involution with respect to the above Poisson structure[1], where $h_g, \cdots, h_{2g}$ are the Casimirs, and $h_0, \cdots, h_{g-1}$ generate the Hamiltonian vector fields $\mathcal{X}_1, \cdots, \mathcal{X}_g$ on M_g by $\mathcal{X}_i := \{\cdot\,, h_{g-i}\}$. Introducing $D(z) := \sum_{i=0}^{g-1} z^i\, \mathcal{X}_{g-i}$, these vector fields can be simultaneously written as follows (see [17, Th. 3.1]):

$$
\begin{aligned}
D(z)u(x) &= 2\,\frac{u(x)v(z)-v(x)u(z)}{x-z},\\
D(z)v(x) &= \frac{w(x)u(z)-u(x)w(z)}{x-z} - u(x)u(z),\\
D(z)w(x) &= 2\Big(\frac{v(x)w(z)-w(x)v(z)}{x-z} + v(x)u(z)\Big).
\end{aligned}
\tag{2.1}
$$

Since Φ_g is submersive and since the above g Hamiltonian vector fields are independent at a generic point of M_g, a simple count shows that the triplet $(M_g, \{\cdot\,,\cdot\}\,, \Phi_g)$ is a (complex) Liouville integrable system.

2.2. **Algebraic integrability.** It was shown by Mumford that $(M_g, \{\cdot\,,\cdot\}\,, \Phi_g)$ is actually an a.c.i. system, which means that, in addition to Liouville integrability, the generic fiber of the momentum map Φ_g is an affine part of an Abelian variety (complex algebraic torus), and that the above Hamiltonian vector fields are constant (translation invariant) on these tori. We sketch the proof, which Mumford attributes to Jacobi. To a polynomial $h(x) \in H_g$, one naturally associates two geometrical objects:

- The *spectral curve* $C_g(h)$ is defined to be a completion of the affine curve in $\mathbf{C}^2$ given by $y^2 = h(x)$ by adding one smooth point ∞. This is an integral projective (possibly singular) hyperelliptic curve of (arithmetic) genus g.
- The *level set* $M_g(h)$ is defined to be the fiber of Φ_g over $h(x)$.

Theorem 2.1 (Mumford)**.** *Suppose that $h(x) \in H_g$ has no multiple roots, so that $C_g(h)$ is an irreducible projective smooth hyperelliptic curve of genus g. Let $J_g(h)$ and $\Theta_g(h)$ be the Jacobian variety and the theta divisor of $C_g(h)$. Then there is an isomorphism $M_g(h) \cong J_g(h) \setminus \Theta_g(h)$ by which the vector fields $\mathcal{X}_1, \dots, \mathcal{X}_g$ are transformed into independent translation invariant vector fields on $J_g(h)$.*

[1]Actually, they are in involution with respect to a whole family of compatible Poisson structures, see [22, Ch. VI (4.4)].

Outline of the proof. One first proves that there is an isomorphism between $M_g(h)$ and an open dense subset

$$\mathcal{S} := \{\sum_{i=1}^{g}[P_i] \in \mathrm{Sym}^g(C_g(h) \setminus \{\infty\}) \mid i \neq j \Rightarrow P_i \neq \imath(P_j)\}$$

of $\mathrm{Sym}^g(C_g(h))$, where $\imath : C_g(h) \to C_g(h)$ is the hyperelliptic involution. This isomorphism is given by

$$\ell(x) = \begin{pmatrix} v(x) & w(x) \\ u(x) & -v(x) \end{pmatrix} \mapsto \sum_{\text{roots } x_i \text{ of } u(x)} [(x_i, v(x_i))] \tag{2.2}$$

when $u(x)$ has no multiple roots, which naturally extends to the whole of $M_g(h)$ by the interpolation formula. The next step is to show that the Abel-Jacobi map induces an isomorphism between $\mathcal{S}$ and $J_g(h) \setminus \Theta_g(h)$. Combined with the first step, this yields the isomorphism between $M_g(h)$ and $J_g(h) \setminus \Theta_g(h)$.

As for the translation invariance of the vector fields $\mathcal{X}_1, \dots, \mathcal{X}_g$ on $J_g(h)$, it suffices to prove that they are translation invariant in the neighborhood of a generic point, because they are holomorphic on $M_g(h)$. We use the above isomorphism to write these Hamiltonian vector fields down in terms of the variables x_i, which yield local coordinates in the neighborhood of a generic point of $\mathcal{S}$. We calculate $D(z)u(x_i)$ in two different ways:

$$\begin{aligned} D(z)u(x)\big|_{x=x_i} &= 2\,\frac{v(x_i)u(z)}{z - x_i} = 2y_i \prod_{k\neq i}(z - x_k), \\ D(z)u(x)\big|_{x=x_i} &= -\prod_{k\neq i}(x_i - x_k)D(z)x_i. \end{aligned}$$

Thus

$$\frac{D(z)x_i}{y_i} = -2\prod_{k\neq i}\frac{z - x_k}{x_i - x_k}.$$

By using the interpolation formula, we obtain

$$\sum_{i=1}^{g} x_i^{j-1}\frac{D(z)x_i}{y_i} = -2\sum_{i=1}^{g} x_i^{j-1}\prod_{k\neq i}\frac{z - x_k}{x_i - x_k} = -2z^{j-1}$$

for $j = 1, \cdots, g$. It follows that in terms of the local coordinates x_i, the vector fields $\mathcal{X}_i$ are expressed by

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_g \\ \vdots & & & \vdots \\ x_1^{g-1} & x_2^{g-1} & \cdots & x_g^{g-1} \end{pmatrix} \begin{pmatrix} \frac{\mathcal{X}_g x_1}{y_1} & \frac{\mathcal{X}_{g-1} x_1}{y_1} & \cdots & \frac{\mathcal{X}_1 x_1}{y_1} \\ \frac{\mathcal{X}_g x_2}{y_2} & \cdots & & \frac{\mathcal{X}_1 x_2}{y_2} \\ \vdots & & & \vdots \\ \frac{\mathcal{X}_g x_g}{y_g} & \cdots & & \frac{\mathcal{X}_1 x_g}{y_g} \end{pmatrix} = -2\,\mathbb{I}_g. \tag{2.3}$$

The g differential forms $\{\sum_{i=1}^{g} x_i^j \mathrm{d}x_i/y_i\}_{j=0,\dots,g-1}$ on $\mathcal{S}$ are seen to be the dual basis to $\{\mathcal{X}_i\}_{i=1,\dots,g}$ (up to a scalar) by (2.3). Since $\{\sum_{i=1}^{g} x_i^j \mathrm{d}x_i/y_i\}_{j=0,\dots,g-1}$ constitute under the Abel-Jacobi map a basis for the space of holomorphic one-forms on $J_g(h)$, it follows that $\mathcal{X}_1, \dots, \mathcal{X}_g$ extends to holomorphic (hence translation invariant) vector fields on $J_g(h)$. $\square$

2.3. **Singular fiber.** We consider what happens in Theorem 2.1 when $C_g(h)$ is singular. For a coherent sheaf $\mathcal{F}$ on $C_g(h)$ and $k \in \mathbf{Z}$, we write $\mathcal{F}(k)$ for $\mathcal{F} \otimes \mathcal{O}_{C_g(h)}(k[\infty])$. For any $h(x) \in H_g$, we define $J_g(h)$ and $\bar{J}_g(h)$ respectively to be the generalized Jacobian variety of $C_g(h)$ (which parametrizes invertible sheaves on $C_g(h)$ of degree zero) and its compactification (which parametrizes torsion free $\mathcal{O}_{C_g(h)}$-modules $\mathcal{L}$ of rank one such that $h^0(C_g(h), \mathcal{L}) - h^1(C_g(h), \mathcal{L}) = 1 - g$) (see [10]). We have $J_g(h) = \bar{J}_g(h)$ if $h(x)$ has no multiple root. We have a natural inclusion $J_g(h) \subset \bar{J}_g(h)$ (see [19]). We also define

$$\Theta_g(h) := \{L \in J_g(h) \mid h^0(C_g(h), L(g-1)) \neq 0\}, \tag{2.4}$$

$$\bar{\Theta}_g(h) := \{\mathcal{L} \in \bar{J}_g(h) \mid h^0(C_g(h), \mathcal{L}(g-1)) \neq 0\}. \tag{2.5}$$

Note that we have $h^0(C_g(h), \mathcal{L}(g-1)) = h^1(C_g(h), \mathcal{L}(g-1))$ for any $\mathcal{L} \in \bar{J}_g(h)$.[2] We set

$$M_g(h)_{reg} := \{l(x) \in M_g(h) \mid l(a) \text{ is regular for all } a \in \mathbb{P}^1\}.$$

(Recall that $A \in M_2(\mathbf{C})$ is *regular* iff all eigenspaces of A are one-dimensional. Note that the matrix $l(\infty) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ of leading coefficients is regular.) When $h(x)$ has no multiple root, we have $M_g(h) = M_g(h)_{reg}$. Here we state a special case of a result of Beauville [8].

Theorem 2.2 (Beauville)**.** *For any $h \in H_g$, there exist isomorphisms*

$$M_g(h) \cong \bar{J}_g(h) \setminus \bar{\Theta}_g(h) \qquad \textit{and} \qquad M_g(h)_{reg} \cong J_g(h) \setminus \Theta_g(h),$$

where the latter is a restriction of the former.

Outline of the proof. Let $f : C_g(h) \to \mathbb{P}^1$ be the map given by $(x, y) \mapsto x$. We take $\mathcal{L} \in \bar{J}_g(h) \setminus \bar{\Theta}_g(h)$. We see that the condition $\mathcal{L} \notin \bar{\Theta}_g(h)$ implies that there exists an isomorphism $E := f_*(\mathcal{L}(g-1)) \cong \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$ of $\mathcal{O}_{\mathbb{P}^1}$-modules (and vice versa), which is unique up to the conjugation by an element of $GL_2(\mathbf{C})$. Once we fix this isomorphism, the map $E \to E(g+1)$ defined by the multiplication by $y \in \mathcal{O}_{C_g(h)}$ is represented by a matrix $\tilde{l}(x) \in M_2(\mathbf{C}[x])$ such that all the entries of $\tilde{l}(x)$ are of degree $\leqslant g+1$. We also have $-\det \tilde{l}(x) = h(x)$ by the Cayley-Hamilton formula. In the $GL_2(\mathbf{C})$-conjugate class of $\tilde{l}(x)$, there exists a unique $l(x)$ which belongs to $M_g(h)$ (cf. [8, (1.5)]). It follows that the correspondence $\mathcal{L} \mapsto l(x)$ defines a bijection $\bar{J}_g(h) \setminus \bar{\Theta}_g(h) \cong M_g(h)$. In order to see this is an isomorphism, we simply notice that the same argument works after any base change. It is shown in [8, (1.11-13)] that the restriction of this isomorphism defines $M_g(h)_{reg} \cong J_g(h) \setminus \Theta_g(h)$. □

Remark 2.3. We briefly explain that the two isomorphisms constructed by Mumford and Beauville coincide when $h(x) \in H_g$ has no multiple root. We take $l(x) = \begin{pmatrix} v(x) & w(x) \\ u(x) & -v(x) \end{pmatrix} \in M_g(h)$. Mumford associates to $l(x)$ the invertible sheaf $L = \mathcal{O}_{C_g(h)}(D - g[\infty])$ where $D = \sum_{i=1}^{g} [(x_i, v(x_i))]$ with $u(x) = \prod_{i=1}^{g} (x - x_i)$. We set $E := f_*(L(g-1)) = f_*(\mathcal{O}_{C_g(h)}(D - [\infty]))$. Then we can choose an isomorphism $E(1) \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$ in such a way that on the

[2] When $\mathcal{L} \in J_g(h)$, this is a consequence of the Riemann-Roch theorem (cf. [21]). For a general $\mathcal{L} \in \bar{J}_g(h)$, the proof can be reduced to the previous case, because one can find a partial normalization $f : C' \to C_g(h)$ and an invertible sheaf L on C' of degree zero such that $f_*(L) = \mathcal{L}$ (cf. [9, p. 101]).

global section $u(x)$ and $y - v(x)$ are mapped to the standard basis of $\mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$. (Note that $\{u(x), y - v(x)\}$ is a basis of $H^0(C_g(h), L(g))$.) Then the multiplication by y is represented by $l(x)$, since $(u(x),\ y - v(x))y = (u(x),\ y - v(x))l(x)$ follows from the relation $y^2 = h(x) = u(x)w(x) + v(x)^2$.

Beauville also showed that the Hamiltonian vector fields are transformed by this isomorphism to the vector fields generated by the group action of $J_g(h)$, but the proof works only when $C_g(h)$ is non-singular. It should be possible to modify his argument to deal with singular cases, but we avoid it. Instead, we limit ourselves to consider a very singular rational curve obtained by taking $h(x) = x^{2g+1}$, so that the curve is given by $y^2 = x^{2g+1}$. For this curve we will make the above isomorphism explicit, which entails in particular an explicit description of $M_g(h)_{reg}$ as a subset of $M_g(h)$, a description of the Jacobian variety as the additive group $\mathbf{C}^g$, and a description of the theta divisor as a subvariety of $\mathbf{C}^g$. The latter two descriptions will be given in the following section. We will then discuss the Hamiltonian vector fields in §4.

3. Generalized Jacobian and its compactification

For a positive integer g, we define C_g to be the (complete, singular) hyperelliptic curve defined by the equation $y^2 = x^{2g+1}$. In this section, we study in detail the structure of the generalized Jacobian of C_g and its compactification.

3.1. **Generalized Jacobian.** Let J_g be the generalized Jacobian variety of C_g, which parametrizes isomorphism classes of invertible sheaves on C_g of degree zero (cf. [21]).

The normalization of C_g is given by $\pi_g : \mathbb{P}^1 \to C_g;\ \pi_g(t) = (t^2, t^{2g+1})$. Let O and ∞ be the points on $\mathbb{P}^1$ whose coordinates are $t = 0$ and ∞ respectively. The images of O and ∞ by π_g are, by abuse of notation, written by the same letter O and ∞. Note that O is the unique singular point on C_g. We write R_g for the local ring $\mathcal{O}_{C_g,O}$ of C_g at O, which we regard as a subring of $S = \mathcal{O}_{\mathbb{P}^1,O} = \mathbf{C}[t]_{(t)}$ (via π_g). The completions of S and R_g are identified with $\mathbf{C}[[t]]$ and $\mathbf{C}[[t^2, t^{2g+1}]]$ respectively. The following isomorphisms play an important role throughout this paper (see, for example, [21]):

$$\mathbf{C}^g \cong \mathbf{C}[[t]]^*/\mathbf{C}[[t^2, t^{2g+1}]]^* \cong S^*/R_g^* \cong \mathrm{Div}^0(C_g \setminus \{O\})/\operatorname{div}(R_g^*) \cong J_g. \tag{3.1}$$

Here the first map is given by

$$\vec{a} = (a_1, \dots, a_g) \mapsto f(t; \vec{a}) := \exp(\sum_{i=1}^{g} a_i t^{2i-1}) \mod \mathbf{C}[[t^2, t^{2g+1}]]^*.$$

The second map is induced by the "inclusion to their completion" $S \subset \mathbf{C}[[t]]$ and $R_g \subset \mathbf{C}[[t^2, t^{2g+1}]]$. The third map associates to the class of $f \in S^*$ its divisor class $\operatorname{div}(f)$. The fourth map is defined by $D \mapsto \mathcal{O}(-D)$, where for $D \in \mathrm{Div}(C_g \setminus \{O\})$ we write the corresponding invertible sheaf by $\mathcal{O}(D)$. We often identify all the five groups appearing in (3.1) altogether.

It is convenient to introduce the following notations:

Definition 3.1. (1) We define polynomials $\chi_n \in \mathbf{C}[a_1, a_2, \dots]$ for $n \in \mathbf{Z}_{\geqslant 0}$ by

$$\exp(\sum_{i=1}^{\infty} a_i t^{2i-1}) = \sum_{n=0}^{\infty} \chi_n t^n \qquad \text{in } \mathbf{C}[[t]]. \tag{3.2}$$

For example, we have $\chi_0 = 1, \chi_1 = a_1, \chi_2 = \frac{a_1^2}{2}, \chi_3 = \frac{a_1^3}{6} + a_2$. One sees that χ_n is a polynomial in the variables $a_1, \dots, a_{[\frac{n+1}{2}]}$. We set $\chi_n = 0$ for $n \in \mathbf{Z}_{<0}$.

(2) We set $f_g(t; \vec{a}) := \sum_{n=0}^{2g-1} \chi_n(\vec{a}) t^n$ for $\vec{a} \in \mathbf{C}^g$. Since $f_g(t; \vec{a}) \equiv f(t; \vec{a})$ in $\mathbf{C}[[t]]^* / \mathbf{C}[[t^2, t^{2g+1}]]^*$, the invertible sheaf $L := \mathcal{O}(-\operatorname{div}(f_g(t; \vec{a}))) \in J_g$ corresponds to $\vec{a} \in \mathbf{C}^g$ by (3.1).

In order to study the structure of J_g, we need to introduce some definitions.

Definition 3.2. For a natural number k, we define the Abel-Jacobi map

$$\mathrm{aj}_{g,k} : \mathrm{Sym}^k(C_g \setminus \{O\}) \to J_g$$

by $\mathrm{aj}_{g,k}(D) := \mathcal{O}(D - k[\infty])$.

Definition 3.3. We define a $(g \times 2g)$-matrix

$$X_{2g} := \begin{pmatrix} \chi_1 & \chi_0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \chi_3 & \chi_2 & \chi_1 & \chi_0 & 0 & 0 & \cdots & 0 \\ \chi_5 & \chi_4 & \chi_3 & \chi_2 & \chi_1 & \chi_0 & \cdots & 0 \\ \vdots & & & & & & & \vdots \\ \chi_{2g-1} & \chi_{2g-2} & \chi_{2g-3} & \chi_{2g-4} & \chi_{2g-5} & \cdots & \chi_1 & \chi_0 \end{pmatrix}$$

with entries in $\mathbf{C}[a_1, \dots, a_g]$. For $0 \leqslant k \leqslant 2g$, let X_k be the $(g \times k)$-submatrix of X_{2g} consisting of the left k columns of X_{2g}.

Lemma 3.4. *Let $k \in \mathbf{Z}$ and $\vec{a} = \mathbf{C}^g$. Recall that $L = \mathcal{O}(-\operatorname{div}(f_g(t; \vec{a})))$ is the corresponding invertible sheaf.*

(1) *Assume that $0 \leqslant k \leqslant 2g - 1$. Then the following conditions are equivalent:*
   (a) $h^0(C_g, L(k)) \neq 0$.
   (b) *There exists $h(t) \in \mathbf{C}[t] \setminus \{0\}$ such that $\deg h(t) \leqslant k$ and $f_g(t; \vec{a}) h(t) \in R_g$.*
   (c) *There exists $\vec{b} = (b_i)_{i=0}^k \in \mathbf{C}^{k+1}$ such that $X_{k+1} \vec{b} = 0$ and $\vec{b} \neq 0$.*

(2) *Assume that $0 \leqslant k \leqslant 2g - 1$. Then the following conditions are equivalent:*
   (a) *L is in the image of $\mathrm{aj}_{g,k}$.*
   (b) *There exists $h(t) \in \mathbf{C}[t]$ such that $\deg h(t) \leqslant k$ and $f_g(t; \vec{a}) h(t) \in R_g^*$.*
   (c) *There exists $\vec{b} = (b_i)_{i=0}^k \in \mathbf{C}^{k+1}$ such that $X_{k+1} \vec{b} = 0$ and $b_0 \neq 0$.*

Proof. First we prove (1):

$$
\begin{aligned}
h^0(C_g, L(k)) \neq 0 \Leftrightarrow & \exists r \in R_g \setminus \{0\},\ \operatorname{div}(r) - \operatorname{div}(f_g(t;\vec{a})) + k \cdot \infty \geqslant 0 \\
\Leftrightarrow & \exists h (= \frac{r}{f_g(t;\vec{a})}) \ \in \mathbf{C}[t] \setminus \{0\},\ \text{s.t.} \deg h(t) \leqslant k,\ f_g(t;\vec{a})h(t) \in R_g \\
\Leftrightarrow & \exists h(t) = \sum_{j=0}^{k} b_j t^j \neq 0 \text{ s.t. } f_g(t;\vec{a})h(t) = \sum_n (\sum_j b_j \chi_{n-j}) t^n \in R_g \\
\Leftrightarrow & \exists \vec{b} = (b_j) \in \mathbf{C}^{k+1} \setminus \{0\} \text{ s.t. } \sum_j b_j \chi_{n-j} = 0 \ (n = 1, 3, \dots, 2g-1) \\
\Leftrightarrow & \exists \vec{b} = (b_j) \in \mathbf{C}^{k+1} \setminus \{0\} \text{ s.t. } X_{k+1}\vec{b} = 0.
\end{aligned}
$$

Next we prove (2). If $E = \sum_{i=1}^{k} [\pi_g(\alpha_i)] \in \operatorname{Sym}^k(C_g \setminus \{O\})$ with $\alpha_i \in \mathbb{P}^1 \setminus \{O\}$, then $\operatorname{aj}_{g,k}(E)$ is represented by $h(t)^{-1} \in S^*$ where $h(t) = \prod_{i=1}^{k}(1 - \frac{t}{\alpha_i})$. (Here the factors with $\alpha_i = \infty$ are regarded as 1. Hence $h(t)$ is a polynomial of degree $\leqslant k$.) Since $\vec{a}$ is represented by $f_g(t;\vec{a}) \in S^*$, we have $\vec{a} = \operatorname{aj}_{g,k}(E) \Leftrightarrow f_g(t;\vec{a})h(t) \in R_g^*$. This proves (a) $\Leftrightarrow$ (b). The equivalence of (b) and (c) is seen in the same way as (1). □

The above lemma justifies the following definition.

Definition 3.5. We define the *theta divisor* Θ_g to be the zero locus of the polynomial $\det(X_g) \in \mathbf{C}[a_1, \dots, a_g]$ in $\mathbf{C}^g$. This is a divisor on $\mathbf{C}^g$, but we identify it with a divisor on J_g via the isomorphism (3.1), which is the same as $\Theta_g(x^{2g+1})$, defined in (2.4).

Corollary 3.6. (1) *For $L \in J_g$, the following conditions are equivalent:*

$$(a)\ L \in \Theta_g, \qquad (b)\ h^0(C_g, L(g-1)) \neq 0, \qquad (c)\ \det(X_g) = 0.$$

(2) *We have* $\operatorname{Im}(\operatorname{aj}_{g,g-1}) \subset \Theta_g$. *However,* $\operatorname{Im}(\operatorname{aj}_{g,g-1}) \neq \Theta_g$ *if* $g \geqslant 3$.
(3) *For any $L \in J_g$ and $k \geqslant g$, the equivalent conditions in Lemma 3.4 (1) hold. However,* $\operatorname{Im}(\operatorname{aj}_{g,g}) \neq J_g$ *if* $g \geqslant 2$.
(4) *The image of* $\operatorname{aj}_{g,g+1}$ *contains* $J_g \setminus \Theta_g$.

Proof. (1) When $k = g - 1$, the condition of Lemma 3.4 (1-c) is rephrased as (c), which proves *(b)* $\Leftrightarrow$ *(c)*. The equivalence of (a) and (c) is the definition of Θ_g.

(2) The first statement follows from a trivial fact $R_g^* \subset R_g$. The second statement is an effect of non-zero elements of $R_g \setminus R_g^*$. A concrete example is given by $g = 3$ and $L = \mathcal{O}(-\operatorname{div}(1 - t^5))$. (See the last line in this proof for the case $g = 1, 2$.)

(3) The first statement follows from the Riemann-Roch theorem. The second statement is an effect of elements of $R_g \setminus R_g^*$. A concrete example is given by $g = 2$ and $L = \mathcal{O}(-\operatorname{div}(1 - t^3))$. (See the last line in this proof for the case $g = 1$.)

(4) We take $L \in J_g \setminus \Theta_g$. Then X_g is of rank g by (1). Hence X_{g+2} is also of rank g, and the linear equation

$$(*) \qquad X_{g+2}\vec{b} = 0$$

has two independent solutions $\vec{b} = (b_i)_{i=0}^{g+1} \in \mathbf{C}^{g+2}$. By Lemma 3.4 (2), it is enough to show that (at least) one of these two solutions satisfies $b_0 \neq 0$. We suppose that there exist two independent solutions to $(*)$ with $b_0 = 0$. Because the first row of $(*)$ reads $a_1 b_0 + b_1 = 0$, we have $b_1 = 0$ as well. Let Y be the lower-right $((g-1) \times g)$-submatrix of X_{g+2}. (This is to say Y is constructed by removing the top row and the left two columns from X_{g+2}.) Then $Y\vec{c} = 0$ has two independent solutions. Hence Y is of rank $\leqslant g-2$. However, since Y is the same as a submatrix of X_g (obtained by removing the bottom row from X_g) this implies X_g is of rank $\leqslant g-1$. This contradicts the fact that the rank of X_g is g.

The implication $b_0 = 0 \Rightarrow b_1 = 0$ shows that the implication (1-c) $\Rightarrow$ (2-c) holds when $k = 2$ in Lemma 3.4. This explains why the equality holds in (2) for $g = 1, 2$, and in (3) for $g = 1$. $\square$

The following lemma gives an explicit formula for the map (3.1), composed with $\mathrm{aj}_{g,g}$, restricted to an open dense subset:

Lemma 3.7. *The composition* $\mathrm{Sym}^g(C_g \setminus \{O, \infty\}) \overset{\mathrm{aj}_{g,g}}{\to} J_g \cong \mathbf{C}^g$ *is described as follows:*

$$D = \sum_{k=1}^{g} [\pi_g(\alpha_k)] \mapsto \vec{a} = (\frac{1}{2i-1} \sum_{k=1}^{g} \alpha_k^{-(2i-1)})_{i=1}^{g} \qquad (\alpha_k \in \mathbb{P}^1 \setminus \{O, \infty\}).$$

Proof. We claim that the formal power series defined by $\Xi(t) := (1-t)\exp(\sum_{j=1}^{\infty} \frac{t^{2j-1}}{2j-1})$ belongs to $\mathbf{C}[[t^2]]$. Indeed, we have

$$\Xi(t) = \exp(\log(1-t) + \sum_{j=1}^{\infty} \frac{t^{2j-1}}{2j-1}) = \exp(-\sum_{i=1}^{\infty} \frac{t^i}{i} + \sum_{j=1}^{\infty} \frac{t^{2j-1}}{2j-1}) = \exp(-\sum_{i=1}^{\infty} \frac{t^{2i}}{2i}),$$

which belongs to $\mathbf{C}[[t^2]]$. Consequently, we have

$$\Xi_g(t) := (1-t)\exp(\sum_{j=1}^{g} \frac{t^{2j-1}}{2j-1}) \in \mathbf{C}[[t^2, t^{2g+1}]].$$

By replacing t by $\alpha_k^{-1} t$ for $k = 1, \dots, g$ and taking a product, we get

$$\prod_{k=1}^{g} \Xi_g(\alpha_k^{-1} t) = \prod_{k=1}^{g} (1 - \alpha_k^{-1} t) \exp(\sum_{i=1}^{g} \frac{(\alpha_k^{-1} t)^{2i-1}}{2i-1}) = h(t) f(t; \vec{a}) \in \mathbf{C}[[t^2, t^{2g+1}]],$$

where $h(t) = \prod_{k=1}^{g} (1 - \alpha_k^{-1} t)$. Now we take $D := \sum_{k=1}^{g} [\pi_g(\alpha_k)] \in \mathrm{Sym}^g(C_g \setminus \{O, \infty\})$. Then $\mathrm{aj}_{g,g}(D)$ is represented by $h(t)^{-1}$ in $\mathbf{C}[[t]]^* / \mathbf{C}[[t^2, t^{2g+1}]]^*$. The above calculation shows that, in $\mathbf{C}[[t]]^* / \mathbf{C}[[t^2, t^{2g+1}]]^*$, the class of $h(t)^{-1}$ is the same as $f(t; \vec{a})$, which represents $\vec{a}$. This completes the proof. $\square$

3.2. **Compactification of the Generalized Jacobian.** We write $\bar{J}_g$ for the compactified Jacobian of C_g which parametrizes isomorphism classes of torsion free $\mathcal{O}_{C_g}$-modules $\mathcal{L}$ of rank one such that $h^0(C_g, \mathcal{L}) - h^1(C_g, \mathcal{L}) = 1 - g$ (see [10, 19]). We have a natural inclusion $J_g \subset \bar{J}_g$, by which we regard J_g as a Zariski dense open subscheme in $\bar{J}_g$ We also define $\bar{\Theta}_g = \{\mathfrak{L} \in \bar{J}_g \mid h^0(C_g, \mathfrak{L}(g-1)) \neq 0\}$. Similarly to Θ_g, we see that $\bar{\Theta}_g$ is the same as $\bar{\Theta}_g(x^{2g+1})$, defined in (2.5).

The normalization $\pi_g : \mathbb{P}^1 \to C_g$ factors as $\mathbb{P}^1 \stackrel{\pi_k}{\to} C_k \stackrel{\pi_{k,g}}{\to} C_g$ for $k = 1, \dots, g$. Explicitly, $\pi_{k,g}$ is given by $\pi_{k,g}(x, y) = (x, x^{g-k}y)$. We have a push-forward $(\pi_{k,g})_* : \bar{J}_k \to \bar{J}_g$. We also have an action of J_g on $\bar{J}_g$ defined by $L \cdot \mathfrak{L} = L \otimes \mathfrak{L}$ for $L \in J_g$ and $\mathfrak{L} \in \bar{J}_g$.

Lemma 3.8. *Let $k \in \{1, \dots, g\}$.*

(1) *The push-forward defines an isomorphism $(\pi_{g-1,g})_* : \bar{J}_{g-1} \to \bar{J}_g \setminus J_g$.*
(2) *For any $L \in J_g$ and $\mathfrak{L} \in \bar{J}_k$, we have $(\pi_{k,g})_*((\pi_{k,g})^* L \cdot \mathfrak{L}) = L \cdot (\pi_{k,g})_* \mathfrak{L}$*
(3) *We have a commutative diagram of algebraic groups*

$$\begin{array}{ccc} \mathbf{C}^g & \stackrel{(3.1)}{\cong} & J_g \\ \downarrow & & \downarrow{\scriptstyle (\pi_{k,g})^*} \\ \mathbf{C}^k & \stackrel{(3.1)}{\cong} & J_k \end{array}$$

where the left vertical map is defined by $(a_i)_{i=1}^g \mapsto (a_i)_{i=1}^k$.
(4) *For any $\mathfrak{L} \in \bar{J}_k$, we have $\mathfrak{L} \in \bar{\Theta}_k$ if and only if $(\pi_{k,g})_* \mathfrak{L} \in \bar{\Theta}_g$.*

Proof. It is proved in [9, Lemma 3.1] that $(\pi_{k,g})_* : \bar{J}_k \to \bar{J}_g$ is a closed embedding. Now (1) follows by induction from the elementary fact that any torsion-free R_g-submodule M of rank one in $\mathbf{C}(C_g) = \mathbf{C}(t)$ satisfies $f(t)M = R_k$ for some $f(t) \in \mathbf{C}(t)^*$ and $k = 0, 1, \dots, g$. (We set $R_0 = S$ by convention.) (2) is a direct consequence of the projection formula. (3) follows from the description of the isomorphism (3.1). Since $\pi_{k,g}$ is a finite map, we have $h^0(C_k, \mathfrak{L}) = h^0(C_g, (\pi_{k,g})_* \mathfrak{L})$, which proves (4). $\square$

4. Singular fiber of the Mumford system with additive degeneration

We use the notations of §2. We apply the results of the previous section to study the level set $M_g(0) := M_g(x^{2g+1})$ of the genus g Mumford system.

4.1. **Matrix realization of the generalized Jacobian.** Let us take $h(x) := x^{2g+1} \in H_g$. Then the spectral curve $C_g(h)$ is precisely C_g considered in the previous section. We write $M_g(0)$ and $M_g(0)_{reg}$ for $M_g(h)$ and $M_g(h)_{reg}$. We define a map

$$i_g : M_{g-1}(0) \to M_g(0) \qquad i_g(l(x)) = xl(x).$$

Lemma 4.1. *Let $l(x) \in M_g(0)$. Then $l(x)$ is in $M_g(0)_{reg}$ iff $l(0) \neq 0$. In other words, we have $M_g(0)_{reg} = M_g(0) \setminus i_g(M_{g-1}(0))$.*

Proof. We first remark that a traceless 2 by 2 matrix A is regular iff $A \neq 0$. Hence $l(x) \in M_g(0)_{reg}$ iff $l(c) \neq 0$ for all $c \in \mathbf{C}$. If $l(c) = 0$ for some $c \in \mathbf{C}$, then $x^{2g+1} = -\det l(x)$ is divisible by $x - c$, thus c must be 0. $\square$

Combined with Theorem 2.2, we obtain

Theorem 4.2. *There exist isomorphisms*

$$\bar{\phi}_g : M_g(0) \cong \bar{J}_g \setminus \bar{\Theta}_g \qquad \textit{and} \qquad \phi_g : M_g(0) \setminus i_g(M_{g-1}(0)) \cong J_g \setminus \Theta_g.$$

Remark 4.3. We give an explicit description of ϕ_g. (Compare with Remark 2.3.) Take $l(x) \in M_g(0) \setminus i_g(M_{g-1}(0))$. Because of the relation $u_0 w_0 + v_0^2 = 0$, we have $u_0 \neq 0$ or $w_0 \neq 0$. In the first case, $l(x)$ is mapped to the invertible sheaf corresponding to the divisor $\sum_{i=1}^g [\alpha_i] - g[\infty]$, where $\alpha_i = v(x_i)/x_i^g$ with $u(x) = \prod_{i=1}^g (x - x_i)$. In the second case, $l(x)$ is mapped to the invertible sheaf corresponding to the divisor $\sum_{j=1}^{g+1} [-\beta_j] - (g+1)[\infty]$, where $\beta_j = v(x_j)/x_j^g$ with $w(x) = \prod_{j=1}^{g+1}(x - x_j)$. Note that for $l(x)$ with $u_0 w_0 \neq 0$, the two definitions give the same divisor class. Indeed, one has

$$\sum_i [\alpha_i] - g[\infty] \equiv -\sum_j [\beta_j] + (g+1)[\infty] \equiv \sum_j [-\beta_j] - (g+1)[\infty],$$

where the first equivalence is seen by $\operatorname{div}(t^{2g+1} - v(t^2)) = \sum_i [\alpha_i] + \sum_j [\beta_j] - (2g+1)[\infty]$, while the second follows from $\operatorname{div}(1 - \frac{t^2}{\gamma^2}) = [\gamma] + [-\gamma] - 2[\infty]$ for any $\gamma \in \mathbf{C} \setminus \{0\}$. We shall consider the inverse map of ϕ_g in §5.1.

4.2. **The Hamiltonian vector fields.**

Theorem 4.4. *The vector fields $\mathcal{X}_1, \dots, \mathcal{X}_g$ on $M_g(0)$ are linearized by the isomorphism $\bar{\phi}_g$ to the vector fields induced by the action of J_g on $\bar{J}_g$. More precisely, we have the following:*

1. *For any $i = 1, \dots, g$, the vector fields $\mathcal{X}_i$ on $M_g(0) \setminus i_g(M_{g-1}(0))$ are mapped to (the restriction of) the invariant vector fields $\frac{\partial}{\partial a_i}$ on $\mathbf{C}^g$ by the isomorphism ϕ_g in Theorem 4.2 composed with* (3.1)*.*
2. *The map $i_g : M_{g-1}(0) \to M_g(0)$ is a closed embedding. The vector fields $\mathcal{X}_1, \dots, \mathcal{X}_{g-1}$ on $M_{g-1}(0)$ are mapped to $\mathcal{X}_1, \dots, \mathcal{X}_{g-1}$ on $M_g(0)$ by i_g, while the vector field $\mathcal{X}_g$ is zero on $i_g(M_{g-1}(0))$.*
3. *The level set $M_g(0)$ is stratified by $g+1$ smooth affine varieties, which are invariant for the flows of the vector fields $\mathcal{X}_1, \dots, \mathcal{X}_g$; they are isomorphic to $\mathbf{C}^k \setminus \Theta_k$ for $k = 0, \dots, g$.*

Proof. (2) follows from (1) and Lemma 3.8. (3) is a consequence of (1) and (2). We prove (1). Since the vector fields in question are all holomorphic, it suffices to show this assertion on some open dense subset. We define

$$\mathcal{S}' := \{\sum_{i=1}^g [(x_i, y_i)] \in \operatorname{Sym}^g(C_g \setminus \{O, \infty\}) \mid x_i \neq x_j \text{ for all } i \neq j\}.$$

Lemma 4.5. *The map $\mathrm{aj}_{g,g}$, restricted to $\mathcal{S}'$, is an open immersion whose image is a dense open subset of J_g.*

Proof. Suppose that $\sum_{i=1}^g [(x_i, y_i)]$ and $\sum_{i=1}^g [(x'_i, y'_i)]$ have the same image in J_g. If we set $\alpha_i = y_i/x_i^g$, $\alpha'_i = y'_i/{x'}_i^g$, then this amounts to saying that $f(t) = \prod_{i=1}^g (1 - \alpha_i^{-1} t)(1 + {\alpha'}_i^{-1} t)$

is in R_g. Since $f(t)$ is of degree $2g$, we must have $f(t)=f(-t)$. This implies $\sum_{i=1}^{g}[\alpha_i] = \sum_{i=1}^{g}[\alpha_i']$ (in $\mathrm{Sym}^g(\mathbb{P}^1 \setminus \{O,\infty\})$) by the definition of $\mathcal{S}'$, and the injectivity follows. The rest follows from Lemma 3.4 (2). $\square$

Now we consider the vector fields on $\mathcal{S}'$. Since the computation made in the proof of Theorem 2.1 is valid in this situation, it follows by putting $x_i=\alpha_i^2, y_i=\alpha_i^{2g+1}$ in (2.3) that, with local coordinates α_i, the vector fields $\mathcal{X}_i$ are expressed by

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1^2 & \alpha_2^2 & \cdots & \alpha_g^2 \\ \vdots & & & \vdots \\ \alpha_1^{2g-2} & \alpha_2^{2g-2} & \cdots & \alpha_g^{2g-2} \end{pmatrix} \begin{pmatrix} \frac{\mathcal{X}_g\alpha_1}{\alpha_1^{2g}} & \frac{\mathcal{X}_{g-1}\alpha_1}{\alpha_1^{2g}} & \cdots & \frac{\mathcal{X}_1\alpha_1}{\alpha_1^{2g}} \\ \frac{\mathcal{X}_g\alpha_2}{\alpha_2^{2g}} & \cdots & & \frac{\mathcal{X}_1\alpha_2}{\alpha_2^{2g}} \\ \vdots & & & \vdots \\ \frac{\mathcal{X}_g\alpha_g}{\alpha_g^{2g}} & \cdots & & \frac{\mathcal{X}_1\alpha_g}{\alpha_g^{2g}} \end{pmatrix} = -\mathbb{I}_g. \tag{4.1}$$

Using Lemma 3.7 and (4.1) one computes that $\mathcal{X}_k a_i = \delta_{i,k}$, for $1 \leqslant i,k \leqslant g$, which leads to (1) in Theorem 4.4.

5. Rational solution to the Mumford system

In view of Theorem 4.4, an explicit description of the inverse map

$$\phi_g^{-1} : J_g \setminus \Theta_g \to M_g(0) \setminus i_g(M_{g-1}(0)); \quad \vec{a} \mapsto \begin{pmatrix} v(x) & w(x) \\ u(x) & -v(x) \end{pmatrix}.$$

of ϕ_g gives rise to a rational solution to the Mumford system. This will be done in §5.1, then we present a concrete algorithm to compute rational solutions in §5.2.

5.1. **The map ϕ_g^{-1}.** We introduce some notations. For $\vec{a}=(a_1,\cdots,a_g) \in \mathbf{C}^g$, let $\bar{X} = \bar{X}(\vec{a})$ be the $2g$ by g matrix:

$$\bar{X} = \begin{pmatrix} \chi_0 & 0 & \cdots & & \\ \chi_1 & 0 & \cdots & & \\ \chi_2 & \chi_0 & 0 & \cdots & \\ \vdots & & & & \\ \chi_g & \chi_{g-2} & \cdots & & \\ \chi_{g+1} & \chi_{g-1} & \cdots & & \\ \vdots & & & & \\ \chi_{2g-2} & \chi_{2g-4} & \cdots & \chi_2 & \chi_0 \\ \chi_{2g-1} & \chi_{2g-3} & \cdots & \chi_3 & \chi_1 \end{pmatrix},$$

where $\chi_i = \chi_i(\vec{a})$ are given by Definition 3.1 (1). We write $\bar{X}_g = \bar{X}_g(\vec{a})$ for the submatrix consisting of the last g rows of $\bar{X}$. We remark that $\bar{X}$ and X_{2g} of Definition 3.3 are closely related (for instance we have $\det \bar{X}_g = \det X_g$), but they come out of different contexts, and it seems more natural to use both of them. We divide $\bar{X}$ into a $g+1$ by g matrix $A=A(\vec{a})$, a $g-1$ by $g-1$ matrix $B=B(\vec{a})$ and a vector $\vec{\phi}=\vec{\phi}(\vec{a}) = {}^t(\phi_1,\cdots,\phi_{g-1})$:

$$\begin{aligned} A_{i,j} &= \bar{X}_{i,j} = \chi_{i-2j+1} \quad \text{for } 1 \leqslant i \leqslant g+1,\ 1 \leqslant j \leqslant g, \\ B_{i,j} &= \bar{X}_{g+1+i,1+j} = \chi_{i-2j+g} \quad \text{for } 1 \leqslant i,j \leqslant g-1, \\ \phi_i &= \bar{X}_{g+1+i,1} = \chi_{i+g} \quad \text{for } 1 \leqslant i \leqslant g-1. \end{aligned}$$

Let $\tau_g = \tau_g(\vec{a})$ be the polynomial function on $\mathbf{C}^g$ given by

$$\tau_g(\vec{a}) = \det \bar{X}_g(\vec{a})(= \det X_g(\vec{a})). \tag{5.1}$$

Note that τ_g is essentially the Schur function associated to the partition $\nu = (g, g-1, \cdots, 1)$. (See (3) in the proof of Proposition 6.1.) Recall that the g vector fields $\mathcal{X}_i$ on $M_g(0)$ induce the translation invariant vector fields $\mathcal{X}_i = \frac{\partial}{\partial a_i}$ on $\mathbf{C}^g$ (Theorem 4.4 (1)). For a rational function $s \in \mathbf{C}(a_1, \cdots, a_g)$ we write $s' := \frac{\partial}{\partial a_1} s$ and $s^{(k)} := \frac{\partial^k}{\partial a_1^k} s$ for $k = 1, 2, \ldots$.

Let U be the open subset of $J_g \setminus \Theta_g = \{\vec{a} \in \mathbf{C}^g \mid \tau_g(\vec{a}) \neq 0\}$ defined by

$$U := \{\vec{a} \in \mathbf{C}^g \mid \det B(\vec{a}) \neq 0 \text{ and } \tau_g(\vec{a}) \neq 0\}.$$

The next proposition is a key to an explicit formula for ϕ_g^{-1}:

Proposition 5.1. *Suppose $\vec{a} \in U$. We denote by $p(t; \vec{a}) = \sum_{k=0}^{g} p_k t^k$ the polynomial, whose coefficients are defined by*

$$\vec{p} = {}^t(p_0, p_1, \cdots, p_g) := A(\vec{a}) \begin{pmatrix} 1 & 0 \\ 0 & -B(\vec{a})^{-1} \end{pmatrix} \begin{pmatrix} 1 \\ \vec{\phi}(\vec{a}) \end{pmatrix}. \tag{5.2}$$

Then $p(t; a)$ is the unique polynomial of degree at most g, which satisfies $p_0 = 1$ and

$$\sum_{k=0}^{2g-1} \chi_k t^k \equiv p(t; \vec{a}) \text{ in } \mathbf{C}[[t]]^* / \mathbf{C}[[t^2, t^{2g+1}]]^*. \tag{5.3}$$

Proof. We see that (5.3) with $p_0 = 1$ is equivalent to the existence of a polynomial $b(t) = 1 + \sum_{j=1}^{g-1} b_j t^{2j}$ such that

$$p(t; \vec{a}) \equiv (\sum_{k=0}^{2g-1} \chi_k t^k) \cdot b(t) \mod t^{2g} \mathbf{C}[[t]]. \tag{5.4}$$

Then we have

$$(5.4) \;\Leftrightarrow\; \bar{X}\vec{b} = \begin{pmatrix} \vec{p} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \;\Leftrightarrow\; \begin{cases} \text{(ia) } A\vec{b} = \vec{p} \\ \text{(ib) } (\vec{\phi}\, B)\vec{b} = {}^t(0, \cdots, 0) \end{cases}$$

where $\vec{b} = {}^t(1, b_1, b_2, \cdots, b_{g-1})$. When $\det B \neq 0$, (ib) has the unique solution

$$\vec{b} = \begin{pmatrix} 1 & 0 \\ 0 & -B^{-1} \end{pmatrix} \begin{pmatrix} 1 \\ \vec{\phi} \end{pmatrix},$$

with which (ia) is equivalent to (5.2). This completes the proof. $\square$

Theorem 5.2. *If $\vec{a} \in U$, then $\phi_g^{-1}(\vec{a}) = \begin{pmatrix} v(x) & w(x) \\ u(x) & -v(x) \end{pmatrix}$ is given by*

$$u(t^2) = \frac{(-1)^g}{p_g(\vec{a})^2} p(t; \vec{a}) p(-t; \vec{a}), \tag{5.5}$$

$$v(x) = \frac{1}{2} \frac{\partial}{\partial a_1} u(x), \qquad w(x) = (x - 2u_{g-1})u(x) - \frac{1}{2} \frac{\partial^2}{\partial a_1^2} u(x). \tag{5.6}$$

For a proof, we need a few lemmas:

Lemma 5.3. *For $k = 1, \cdots, g$. we have*

$$(1)\ \frac{\partial}{\partial a_k}\chi_j = \chi_{j-2k+1}, \qquad (2)\ \frac{\partial}{\partial a_k}\chi_j = \left(\frac{\partial}{\partial a_1}\right)^{2k-1}\chi_j.$$

Proof. By operating with $\frac{\partial}{\partial a_k}$ on (3.2), we obtain

$$t^{2k-1}\exp\left(\sum_{i=1}^{g} a_i t^{2i-1}\right) = \sum_{j=0}^{\infty}\left(\frac{\partial}{\partial a_k}\chi_j\right)t^j.$$

Thus we get $\sum_{j=0}^{\infty}(\frac{\partial}{\partial a_k}\chi_j - \chi_{j-2k+1})t^j = 0$, and (1) follows. (2) follows from (1). □

Lemma 5.4. *For $\vec{a} \in U$, we have the following:*

$$(1)\ \tau_g(\vec{a}) = p_g \det B, \qquad (2)\ \tau_g'(\vec{a}) = p_{g-1}\det B, \qquad (3)\ \tau_g''(\vec{a}) = 2p_{g-2}\det B.$$

Proof. (1) Since $\vec{a} \in U$, we can write $B(\vec{a})^{-1} = \frac{1}{\det B(\vec{a})}\bar{B}(\vec{a})$ where $\bar{B}(\vec{a})$ is the matrix of cofactors of $B(\vec{a})$. We write B_k for the $g-1$ by $g-2$ submatrix of $B(\vec{a})$ obtained by removing the k-th column of $B(\vec{a})$. We have

$$p_g \det B = \chi_g \det B - \sum_{k=1}^{[\frac{g}{2}]}\chi_{g-2k}\sum_{j=1}^{g-1}\bar{B}_{k,j}\phi_j,$$

$$\tau_g = \det \bar{X}_g = \chi_g \det B + \sum_{k=1}^{[\frac{g}{2}]}(-1)^k\chi_{g-2k}\det\left(\vec{\phi}B_k\right).$$

Now the claim follows from the following fact

$$\det\left(\vec{\phi}B_k\right) = (-1)^{k-1}\sum_{j=1}^{g-1}\bar{B}_{k,j}\phi_j. \tag{5.7}$$

(2) Using Lemma 5.3, we get

$$\tau_g' = \det\begin{pmatrix}\chi_{g-1} & \chi_{g-3} & \chi_{g-5} & \cdots \\ \vec{\phi} & & B & \end{pmatrix}.$$

On the other hand, we have

$$p_{g-1}\det B = \chi_{g-1}\det B - \sum_{k=1}^{[\frac{g-1}{2}]}\chi_{g-1-2k}\sum_{j=1}^{g-1}\bar{B}_{k,j}\phi_j,$$

which coincides with τ_g' by (5.7).

(3) Using Lemma 5.3, we have

$$\tau_g'' = \det\begin{pmatrix}\chi_{g-2} & \chi_{g-4} & \chi_{g-6} & \cdots \\ \chi_{g+1} & \chi_{g-1} & \chi_{g-3} & \cdots \\ \chi_{g+2} & \chi_g & \chi_{g-2} & \cdots \\ \vdots & & & \\ \chi_{2g-1} & \chi_{2g-3} & \chi_{2g-5} & \cdots\end{pmatrix} + \det\begin{pmatrix}\chi_{g-1} & \chi_{g-3} & \chi_{g-5} & \cdots \\ \chi_{g} & \chi_{g-2} & \chi_{g-4} & \cdots \\ \chi_{g+2} & \chi_g & \chi_{g-2} & \cdots \\ \vdots & & & \\ \chi_{2g-1} & \chi_{2g-3} & \chi_{2g-5} & \cdots\end{pmatrix}.$$

By using (5.7), the first term in r.h.s. turns out to be $p_{g-2}\det B$. On the other hand, it follows from the following lemma that the first and second terms coincide. This completes the proof. □

Lemma 5.5. *Let $X_0,\dots,X_{2g-1}$ be independent variables. We define two elements in the polynomial ring $\mathbf{C}[X_0,\dots,X_{2g-1}]$:*

$$Q_1 := \det\begin{pmatrix} X_{g-2} & X_{g-4} & X_{g-6} & \cdots \\ X_{g+1} & X_{g-1} & X_{g-3} & \cdots \\ X_{g+2} & X_g & X_{g-2} & \cdots \\ \vdots & & & \\ X_{2g-1} & X_{2g-3} & X_{2g-5} & \cdots \end{pmatrix},\quad Q_2 := \det\begin{pmatrix} X_{g-1} & X_{g-3} & X_{g-5} & \cdots \\ X_{g} & X_{g-2} & X_{g-4} & \cdots \\ X_{g+2} & X_g & X_{g-2} & \cdots \\ \vdots & & & \\ X_{2g-1} & X_{2g-3} & X_{2g-5} & \cdots \end{pmatrix}.$$

Then we have $Q_1 = Q_2$.

Proof. We define a derivation ∂ on $\mathbf{C}[X_0,\dots,X_{2g-1}]$ by $\partial X_j = X_{j-2}$ for $2 \leqslant j \leqslant 2g-1$ and $\partial X_0 = \partial X_1 = 0$. We define

$$T := \det\begin{pmatrix} X_g & X_{g-2} & \cdots & 0 \\ X_{g+1} & X_{g-1} & \cdots & 0 \\ X_{g+2} & X_g & & \vdots \\ \vdots & \vdots & & X_0 \\ X_{2g-1} & X_{2g-3} & \cdots & X_1 \end{pmatrix}.$$

We calculate ∂T in two ways. By differentiating columns, we see that $\partial T = 0$ since $\partial X_0 = \partial X_1 = 0$. By differentiating rows, we see that $\partial T = Q_1 - Q_2$. This completes the proof. $\square$

Proof of Theorem 5.2. From Lemma 5.4 (1) we have $p_g \neq 0$ on U, thus $p(t;\vec{a})$ is written as $p(t;\vec{a}) = \prod_{j=1}^g (1 - \frac{t}{\alpha_j})$ so that $p_g = (-1)^g \prod_{j=1}^g \frac{1}{\alpha_j}$. Proposition 5.1 shows that $u(x) = \prod_{j=1}^g (x - \alpha_j^2)$ (cf. Remark 4.3). Thus we have

$$u(t^2) = \prod_{j=1}^g (t-\alpha_j)(t+\alpha_j) = (\prod_{j=1}^g -\alpha_j^2) p(t;\vec{a}) p(-t;\vec{a}),$$

and (5.5) follows. The action of $\mathfrak{X}_g$ (2.1) on M_g is written as follows:

$$\begin{aligned} \mathfrak{X}_1 u(x) &= 2v(x), \\ \mathfrak{X}_1 v(x) &= -w(x) + (x - u_{g-1} + w_g) u(x), \\ \mathfrak{X}_1 w(x) &= 2(x - u_{g-1} + w_g) v(x). \end{aligned} \tag{5.8}$$

To obtain $v(x)$ and $w(x)$, we use the first two equations, the relation $\mathfrak{X}_1 = \frac{\partial}{\partial a_1}$ which comes from Theorem 4.4 (1), and the fact that $w_g = -u_{g-1}$ on $M_g(0)$, as follows from $u(x)w(x) + v(x)^2 = x^{2g+1}$. $\square$

5.2. **Algorithm.** We present an explicit algorithm to compute a rational solution to the Mumford system. This can be considered as a degenerate version of [15] (see also [17, §10]), where a solution is given in terms of the hyperelliptic $\wp$-function. The function ρ_g defined in (5.9) below corresponds to a degenerate version of the hyperelliptic $\wp$-function.

Definition 5.6. We define a family of polynomials $U_0,\dots,U_{g-1}, V_0,\dots,V_{g-1}, W_0,\dots,W_g \in \mathbf{C}[T_0,\dots,T_{2g}]$ as follows. We set

$$U_{g-1} = T_0, \quad V_{g-1} = \frac{1}{2} T_1, \quad W_g = -T_0.$$

Assume we have defined $U_{g-i}, V_{g-i}, W_{g-i+1}$ for $i = 1, \ldots, k$. Then we define

$$\begin{aligned}
U_{g-k-1} &= \frac{1}{4}\ddot{U}_{g-k} + U_{g-1}U_{g-k} - \frac{1}{2}\Big(\sum_{j=g-k}^{g-1} U_j W_{2g-j-k} + \sum_{j=g+1-k}^{g-1} V_j V_{2g-j-k}\Big), \\
V_{g-k-1} &= \frac{1}{2}\dot{U}_{g-k-1}, \\
W_{g-k} &= -\frac{1}{4}\ddot{U}_{g-k} - U_{g-1}U_{g-k} - \frac{1}{2}\Big(\sum_{j=g-k}^{g-1} U_j W_{2g-j-k} + \sum_{j=g+1-k}^{g-1} V_j V_{2g-j-k}\Big).
\end{aligned}$$

Here $F \mapsto \dot{F}$ is the derivation on $\mathbf{C}[T_0, \ldots, T_{2g}]$ defined by $\dot{T}_i = T_{i+1}$ for $i = 0, 1, \ldots, 2g-1$, and by $\dot{T}_{2g} = 0$. The first examples of U_k are given by

$$\begin{aligned}
U_{g-1} &= T_0, \\
U_{g-2} &= \frac{1}{4}T_2 + \frac{3}{2}T_0^2, \\
U_{g-3} &= \frac{1}{16}T_4 + \frac{5}{8}T_1^2 + \frac{5}{4}T_0T_2 + \frac{5}{2}T_3.
\end{aligned}$$

Proposition 5.7. *Let $\rho_g = \rho_g(\vec{a})$ be the rational function in $\mathbf{C}[a_1, \ldots, a_g, \frac{1}{\tau_g}]$ given by*

$$\rho_g(\vec{a}) = \frac{\partial^2}{\partial a_1^2} \log \tau_g(\vec{a}). \tag{5.9}$$

Then, the functions

$$u_k := U_k(\rho_g, \rho_g', \ldots, \rho_g^{(2g)}), \;\; v_k := V_k(\rho_g, \rho_g', \ldots, \rho_g^{(2g)}), \;\; w_k := W_k(\rho_g, \rho_g', \ldots, \rho_g^{(2g)})$$

give a rational solution for the genus g Mumford system.

Proof. By using Theorem 5.2, when $\vec{a} \in U$, u_{g-1} is written in terms of p_j as

$$u_{g-1} = \frac{2p_{g-2}p_g - p_{g-1}^2}{p_g^2}.$$

From Lemma 5.4, this turns out to be

$$u_{g-1} = \frac{\tau_g''\tau_g - (\tau_g')^2}{\tau_g^2} = \rho_g.$$

Since ρ_g has poles only on Θ_g, the domain of the solution of u_{g-1} is extended from the open subset U of $J_g \setminus \Theta_g$ to $J_g \setminus \Theta_g$.

The first two equations of (5.8) yield

$$\begin{aligned}
v_{g-k} &= \frac{1}{2}u_{g-k}', \\
\frac{1}{2}u_{g-k}'' &= -w_{g-k} + u_{g-k-1} + (w_g - u_{g-1})u_{g-k}.
\end{aligned}$$

If we look at the coefficient of x^{2g-k} in the equation $u(x)w(x) + v(x)^2 = x^{2g+1}$, we get

$$\sum_{j=g-k}^{g-1} u_j w_{2g-j-k} + u_{g-k-1} + w_{g-k} + \sum_{j=g+1-k}^{g-1} v_j v_{2g-j-k} = 0.$$

The proposition follows from these three equations. □

Example 5.8. (Rational solution)
(i) $g = 2$ case:

$$\tau_2(\vec{a}) = \frac{a_1^3}{3} - a_2, \qquad \rho_2(\vec{a}) = \frac{-3a_1(a_1^3 + 6a_2)}{(a_1^3 - 3a_2)^2}.$$

(ii) $g = 3$ case:

$$\tau_3(\vec{a}) = \frac{a_1^6}{45} - \frac{a_1^3 a_2}{3} - a_2^2 + a_1 a_3,$$
$$\rho_3(\vec{a}) = \frac{-3(2a_1^{10} + 675a_1^4 a_2^2 - 1350a_1 a_2^3 - 270a_1^5 a_3 + 675a_3^2)}{(a_1^6 - 15a_1^3 a_2 - 45a_2^2 + 45a_1 a_3)^2}.$$

(iii) $g = 4$ case:

$$\tau_4(\vec{a}) = \frac{a_1^{10}}{4725} - \frac{a_1^7 a_2}{105} - a_1 a_2^3 + \frac{a_1^5 a_3}{15} + a_1^2 a_2 a_3 - a_3^2 - \frac{a_1^3 a_4}{3} + a_2 a_4.$$

6. Relation to the KdV hierarchy

As we already pointed out in the introduction, the Mumford systems and the KdV hierarchy are intimately related. We briefly examine the relationship, with focus on the rational solutions. Recall that the KdV hierarchy is defined by the family of compatible Lax equations (see [16, 20] and references therein):

$$\frac{\partial}{\partial x_{2i-1}} \mathfrak{L} = [\mathfrak{L}_+^{i-\frac{1}{2}}\ ,\ \mathfrak{L}], \quad \text{for } i = 1, 2, 3, \cdots .$$

Here $\mathfrak{L}$ is a differential operator of the form $\partial_x^2 + f$ where f is a function of $\vec{x} = (x, x_1, x_3, \cdots) \in \mathbf{C}^\infty$ and $\partial_x f = \frac{\partial f}{\partial x} + f \cdot \partial_x$. The square root $\mathfrak{L}^{\frac{1}{2}}$ is computed in the ring of formal pseudo-differential operators; the index $+$ in $\mathfrak{L}_+^{i-\frac{1}{2}}$ means that we take the differential part of $\mathfrak{L}^{i-\frac{1}{2}}$. The first three equations ($i = 1, 2, 3$) are given as follows:

$$\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x},$$
$$\frac{\partial f}{\partial x_3} = \frac{1}{4}\frac{\partial^3 f}{\partial x^3} + \frac{3}{2} f \cdot \frac{\partial f}{\partial x},$$
$$\frac{\partial f}{\partial x_5} = \frac{1}{16}\frac{\partial^5 f}{\partial x^5} + \frac{5}{8} f \cdot \frac{\partial^3 f}{\partial x^3} + \frac{5}{4}\frac{\partial f}{\partial x} \cdot \frac{\partial^2 f}{\partial x^2} + \frac{15}{8} f^2 \cdot \frac{\partial f}{\partial x}.$$

In the sequel, we identify x with x_1, as suggested by the first equation of the above list and we consider the rational solutions to the KdV hierarchy. According to [6], there is for every positive integer g an essentially unique solution, depending on g parameters:

(1) Suppose $f = f(x_1, x_3, \dots)$ be a non-zero rational function satisfying the KdV hierarchy. Then there exist $g \in \mathbf{Z}_{>0}$ and $c_1 \in \mathbf{C}$ such that $f(x_1, 0, 0, \dots) = -\frac{g(g+1)}{(x_1 - c_1)^2}$. Moreover, f depends on the g variables $x_1, x_3, \dots, x_{2g-1}$, and is independent of the other variables x_{2i-1}. In this case, we call f a genus g rational solution.
(2) If f and $\tilde{f}$ are genus g rational solutions, then there exist $c_1, c_3, \dots, c_{2g-1} \in \mathbf{C}$ such that $\tilde{f}(x_1 - c_1, \dots, x_{2g-1} - c_{2g-1}) = f(x_1, \dots, x_{2g-1})$.

An explicit formula for these rational solutions is given in the following proposition, which is known in different forms, as indicated in the proof below. The upshot, in connection with our result, is that the rational solutions to the KdV hierarchy of genus $0, 1, \ldots, g$ fill up a very specific invariant manifold of the genus g Mumford system and form, combined, the complement of the completed theta divisor of the compactified Jacobian of the singular curve $y^2 = x^{2g+1}$.

Proposition 6.1. *The function $f = 2\rho_g(\vec{a})$, defined in* (5.9), *gives a rational solution for the KdV hierarchy upon substituting*

$$a_i = x_{2i-1}, \quad \text{for } i = 1, \cdots, g. \tag{6.1}$$

This solution is non-trivial for the first $g-1$ vector fields $\frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_5}, \ldots, \frac{\partial}{\partial x_{2g-1}}$ of the hierarchy, and trivial for the higher ones.

Proof. We sketch three different approaches to this result.

(1) The KdV hierarchy is known to have Wronskian solutions, constructed as follows (See [13] for details): Fix $g \in \mathbf{Z}_{>0}$, and consider g functions $f_1, \ldots, f_g$ of $\vec{x} = (x_1, x_3, x_5, \cdots) \in \mathbf{C}^\infty$; for $k \in \mathbf{Z}_{>0}$ we denote $f_i^{(k)} := \frac{\partial^k}{\partial x_1^k} f_i$. If these functions satisfy

$$\frac{\partial}{\partial x_{2k-1}} f_i = \left(\frac{\partial}{\partial x_1}\right)^{2k-1} f_i, \qquad \text{for } k \in \mathbf{Z}_{>0}, \tag{6.2}$$

then $2\frac{\partial^2}{\partial x_1^2} \log T(\vec{x})$ satisfies the KdV hierarchy, where $T(\vec{x})$ is defined by

$$T(\vec{x}) := \det \begin{pmatrix} f_1 & f_1^{(1)} & \cdots & f_1^{(g-1)} \\ f_2 & f_2^{(1)} & \cdots & f_2^{(g-1)} \\ \vdots & \vdots & & \vdots \\ f_g & f_g^{(1)} & \cdots & f_g^{(g-1)} \end{pmatrix}. \tag{6.3}$$

In view of Lemma 5.3, the functions $f_i := \chi_{2g-2i+1}$ with $a_i = x_{2i-1}$ for $i = 1, \cdots, g$, satisfy (6.2). With this choice of functions, $T(\vec{x})$ is precisely $\tau_g(\vec{a})$, and the result follows.

(2) In [17, IIIa §10-11], Mumford shows, building upon the work [15] of McKean-van Moerbeke that a solution to the Mumford system, associated to an arbitrary *smooth* hyperelliptic curve, yields a solution to the KdV hierarchy. In our case the hyperelliptic curve is not smooth, yet Mumford's argument depends only on (differential) algebra, so we can construct as in the smooth case a rational solution f to the KdV equation from the rational solution which we constructed to the genus g Mumford system. Finally we obtain $f = 2u_{g-1}$, which leads precisely to the proposed solution.

(3) In the Grassmannian approach to the KdV equation [20], to each point of the Sato (universal) Grassmannian one associates a tau function, whose second logarithmic derivative yields a solution to the KdV hierarchy. [3] If one takes the point of the Sato Grassmannian corresponding to the partition $\nu = (g, g-1, \cdots, 1)$, then the associated tau

[3]Precisely, this yields a solution to the KP hierarchy in general; it is a solution to the KdV hierarchy iff it depends only on the odd-indexed variables.

function is given by the Schur function F_ν of ν (cf. [20, §8]). By the very definition (5.1), we have an identity $\tau_g(\vec{a}) = (-1)^{\frac{g(g+1)}{2}} F_\nu(a_1, 0, a_2, 0, a_3, \cdots)$, where F_ν is considered as a function in $t_1, t_2, \cdots$ through [20, (8.4)]. Thus, our function τ_g, which shares the same second logarithmic derivative with F_ν, yields a rational solution to the KdV hierarchy. □

Remark 6.2. In [20, p. 47-48], a relation between the Sato Grassmannian and the compactified generalized Jacobian $\bar{J}_g$ is discussed. To be more precise, we introduce the Grassmannian Gr_g of g-dimensional subspaces W of $\mathbf{C}[t]/(t^{2g}) (\cong \mathbf{C}^{2g})$ satisfying $t^2 W \subset W$. (One can consider Gr_g as a subvariety of the usual Grassmannian $\mathrm{Gr}(g, 2g)$ or of the Sato Grassmannian.) Then Gr_g admits a cell decomposition $\mathrm{Gr}_g = \sqcup_{k=0}^g \mathrm{Gr}_g^{(k)}$ with $\mathrm{Gr}_g^{(k)} \cong \mathbf{C}^k$ $(k = 0, 1, \cdots, g)$, and there exists a bijective morphism $\nu_g : \mathrm{Gr}_g \to \bar{J}_g$ satisfying $\nu_g(\mathrm{Gr}_g^{(k)}) = (\pi_{k,g})_*(J_k)$ for all $k = 0, 1, \cdots, g$ (cf. Lemma 3.8). In particular, the cell decomposition of the Grassmannian corresponds to the stratification of the zero level set $M_g(0)$ of the genus g Mumford system. Note that ν_g is not an isomorphism already in the case $g = 1$. It is conjectured in [20] that ν_g gives the normalization of $\bar{J}_g$.

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