TRANSVERSE POISSON STRUCTURES: THE SUBREGULAR AND MINIMAL ORBITS

P. A. DAMIANOU

Department of Mathematics and Statistics, University of Cyprus
P.O. Box 20537, 1678 Nicosia, Cyprus
E-mail: damianou@ucy.ac.cy

H. SABOURIN

Laboratoire de Mathematiques, UMR 6086 du CNRS, Université de Poitiers
86962 Futuroscope Chasseneuil Cedex France
E-mail herve.sabourin@math.univ-poitiers.fr

P. VANHAECKE

Laboratoire de Mathematiques, UMR 6086 du CNRS, Université de Poitiers
86962 Futuroscope Chasseneuil Cedex France
E-mail pol.vanhaecke@math.univ-poitiers.fr

We study the transverse Poisson structures to adjoint orbits in complex simple Lie algebras with special emphasis on the subregular and minimal orbits.

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1. Some general results

In this first section we prove some general results on the transverse Poisson structures to coadjoint orbits. In the final two sections we specialize to the two extreme and most interesting cases, i.e. the subregular and minimal orbits. With the exception of the results on the minimal orbit which are new we only give the statements of the theorems and we refer to[5] for detailed proofs.

1.1. *Transverse Poisson Structures to adjoint orbits*

The definition of the transverse Poisson structure to a symplectic leaf in a Poisson manifold goes back to A.Weinstein.[8] It is given in the following splitting theorem:

Theorem 1.1 (A. Weinstein, 1983). *Let x_0 be a point in a Poisson manifold M. Then near x_0, M is isomorphic to a product $S \times N$ where S is the symplectic leaf of M, passing through x_0 and N is a submanifold of M transverse to S at x_0, which inherits a Poisson structure from M vanishing at x_0. This Poisson structure on N is called the transverse Poisson structure at x_0.*

When M is the dual $\mathfrak{g}^*$ of a complex Lie algebra $\mathfrak{g}$, equipped with its standard Lie-Poisson structure, we know that the symplectic leaf through $\mu \in \mathfrak{g}^*$ is the co-adjoint orbit $\mathbf{G} \cdot \mu$ of the adjoint Lie group $\mathbf{G}$ of $\mathfrak{g}$. In this case, a natural transverse slice to $\mathbf{G} \cdot \mu$ is obtained in the following way: we choose any complement $\mathfrak{n}$ to the centralizer $\mathfrak{g}(\mu)$ of μ in $\mathfrak{g}$ and we take N to be the affine subspace $\mu + \mathfrak{n}^\perp$ of $\mathfrak{g}^*$. Since $\mathfrak{g}(\mu)^\perp = \mathrm{ad}^*_{\mathfrak{g}}\,\mu$ we have

$$T_\mu(\mathfrak{g}^*) = T_\mu(\mathbf{G} \cdot \mu) \oplus T_\mu(N),$$

so that N is indeed a transverse slice to $\mathbf{G} \cdot \mu$ at μ. Furthermore, defining on $\mathfrak{n}^\perp$ any system of linear coordinates $(q_1, \ldots, q_k)$, and using the explicit formula for Dirac reduction, one can write down explicit formulas for the Poisson matrix $\Lambda_N := (\{q_i, q_j\}_N)_{1 \le i,j \le k}$ of the transverse Poisson structure, from which it follows easily that the coefficients of Λ_N are actually rational functions in $(q_1, \ldots, q_k)$. As a corollary, in the Lie-Poisson case, the transverse Poisson structure is always rational.

One immediately wonders in which cases the Poisson structure on N is polynomial; more precisely, for which Lie algebras $\mathfrak{g}$, for which co-adjoint orbits, and for which complements $\mathfrak{n}$.

In 1989 P. A. Damianou[4] made the conjecture that in gl_n, for a specific choice of slice (orthogonal to the orbit with respect to the Killing form) the transverse Poisson structure is always polynomial. The conjecture was verified for all nilpotent orbits of gl_n, for $n \le 7$ and was proved for some special cases i.e. subregular and minimal orbits. In 2002 R. Cushman and M. Roberts[3] proved that there exists for any nilpotent adjoint orbit of a semi-simple Lie algebra a special choice of a complement $\mathfrak{n}$ such that the corresponding transverse Poisson structure is polynomial. In 2005 H. Sabourin in[6] gave a more general class of complements where the transverse structure is polynomial, using in an essential way the machinery of semi-simple Lie algebras. In this paper the transverse slice is always chosen to lie in the class of complements prescribed by Sabourin.

1.2. *Reduction to nilpotent orbits*

It turns out that the transverse Poisson structure to any adjoint orbit $\mathbf{G} \cdot x$ of a semi-simple (or reductive) algebra $\mathfrak{g}$ is essentially determined by the transverse Poisson structure to the underlying nilpotent orbit $\mathbf{G}(s) \cdot e$ defined by its Jordan decomposition $x = s + e$. In fact we have the following result:[5]

Theorem 1.2. *Let $x \in \mathfrak{g}$ be any element, $\mathbf{G} \cdot x$ its adjoint orbit and $x = s + e$ its Jordan-Chevalley decomposition. Given any complement $\mathfrak{n}_e$ of $\mathfrak{g}(x)$ in $\mathfrak{g}(s)$ and putting $\mathfrak{n} := \mathfrak{n}_s \oplus \mathfrak{n}_e$, where $\mathfrak{n}_s = \mathfrak{g}(s)^\perp$, the parallel affine spaces $N_x := x + \mathfrak{n}^\perp$ and $N := e + \mathfrak{n}^\perp$ are respectively transverse slices to the adjoint orbit $\mathbf{G} \cdot x$ in $\mathfrak{g}$ and to the nilpotent orbit $\mathbf{G}(s) \cdot e$ in $\mathfrak{g}(s)$. The translation which sends N_x to N realises an isomorphism between the transverse Poisson structure on N_x and N. The Poisson structure on both transverse slices is given by the same Poisson matrix in terms of the same affine coordinates restricted to the corresponding transverse slice.*

1.3. *The polynomial and quasi-homogeneous character of the tranverse Poisson structure*

The next general result is that the transverse Poisson structure is a quasi-homogeneous Poisson structure of degree -2 with respect to a set of weights that arise from the representation theory of the corresponding simple Lie algebra. We begin with the definition of quasi-homogeneous Poisson structure; see e.g.[1]

Let $\nu = (\nu_1, \ldots, \nu_d)$ be non-negative integers. A polynomial $P \in \mathbf{C}[x_1, \ldots, x_d]$ is said to be *quasi-homogeneous* (relative to ν) if for some integer κ,

$$\forall t \in \mathbf{C}, P(t^{\nu_1} x_1, \ldots, t^{\nu_d} x_d) = t^\kappa P(x_1, \ldots, x_d).$$

The integer κ is then called the *quasi-degree* of P, denoted by $\varpi(P)$. Similarly, a polynomial Poisson structure $\{\cdot,\cdot\}$ on $\mathbf{C}[x_1, \ldots, x_d]$ is said to be *quasi-homogeneous* (relative to ν) if there exists $\kappa \in \mathbf{Z}$ such that, for every quasi-homogeneous polynomials F and G, their Poisson bracket $\{F, G\}$ is quasi-homogeneous of degree

$$\varpi(\{F, G\}) = \varpi(F) + \varpi(G) + \kappa;$$

Before stating the result, we need some notions from the representation theory of simple Lie algebras. First, we choose a Cartan subalgebra $\mathfrak{h}$ of the semi-simple Lie algebra $\mathfrak{g}$, with corresponding root system $\Delta(\mathfrak{h})$, from which a basis $\Pi(\mathfrak{h})$ of simple roots is selected. Let $\mathcal{O}$ be a nilpotent orbit. According to the Jacobson-Morosov-Kostant correspondence, there exists a canonical triple (h, e, f) of elements of $\mathfrak{g}$, associated with $\mathcal{O}$ and completely determined by the following properties:

(1) (h, e, f) is a $\mathfrak{sl}_2$-triple, i.e., $[h, e] = 2e$, $[h, f] = -2f$ and $[e, f] = h$;
(2) h is the characteristic of $\mathcal{O}$, i.e., $h \in \mathfrak{h}$ and $\alpha(h) \in \{0, 1, 2\}$ for every simple root $\alpha \in \Pi(\mathfrak{h})$.
(3) $\mathcal{O} = \mathbf{G} \cdot e$.

The triple (h, e, f) leads to two decompositions of $\mathfrak{g}$:

(1) A decomposition of $\mathfrak{g}$ into eigenspaces relative to ad_h. Each eigenvalue being an integer we have

$$\mathfrak{g} = \bigoplus_{i \in \mathbf{Z}} \mathfrak{g}(i),$$

where $\mathfrak{g}(i)$ is the eigenspace of ad_h that corresponds to the eigenvalue i. For example, $e \in \mathfrak{g}(2)$ and $f \in \mathfrak{g}(-2)$.

(2) Let $\mathfrak{s}$ be the Lie subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2$, which is generated by h, e and f. The Lie algebra $\mathfrak{g}$ is an $\mathfrak{s}$-module, hence it decomposes as

$$\mathfrak{g} = \bigoplus_{j=1}^{k} V_{n_j},$$

where each V_{n_j} is a simple $\mathfrak{s}$-module, with $n_j + 1 = \dim V_{n_j}$ and with ad_h-weights $n_j, n_j - 2, n_j - 4, \ldots, -n_j$.

Let $\mathcal{N}_h$ be the set of ad_h-invariant complements to $\mathfrak{g}(e)$.

Theorem 1.3. *Let g be a semi-simple Lie algebra, let $\mathcal{O}$ be a nilpotent orbit of $\mathfrak{g}$ with canonical triple (h, e, f), and let $\mathfrak{n}$ be in $\mathcal{N}_h$. The transverse Poisson structure on $N := e + \mathfrak{n}^{\perp}$ is a polynomial Poisson structure that is quasi-homogeneous of degree -2, relative to the quasi-degrees $n_1 + 2, \ldots, n_k + 2$, where $n_1, \ldots, n_k$ denote the highest weights of $\mathfrak{g}$ as an $\mathfrak{s}$-module.*

A transverse Poisson structure given by theorem 1.3 will be called an adjoint transverse Poisson structure, or ATP-structure.

2. The subregular case

We will give an explicit description of the Transverse Poisson structure in the case of the subregular orbit $\mathcal{O}_{sr} \subset \mathfrak{g}$, where $\mathfrak{g}$ is a semi-simple Lie algebra. Recall that an element Z in $\mathfrak{g}$ is subregular if $\dim \mathfrak{g}(Z) = Rk(\mathfrak{g})+2$. In this case, the generic rank of the ATP-structure on N is 2 and we know $\dim N - 2$ independent Casimirs, namely the basic Ad-invariant functions on $\mathfrak{g}$, restricted to N. It follows that the ATP-structure is the determinantal structure (also called Nambu structure), determined by these Casimirs, up to multiplication by a function. What is much less trivial to show is that this function is actually just a non-zero *constant*. For this we will use Brieskorn's theory of simple singularities.

2.1. *Invariant functions and Casimirs*

Let $\mathcal{O}_{sr} = \mathbf{G} \cdot e$, be a subregular orbit in the semi-simple Lie algebra $\mathfrak{g}$ of rank ℓ, let (h, e, f) be the corresponding canonical $\mathfrak{sl}_2$-triple and consider the transverse slice $N := e+\mathfrak{n}^{\perp}$ to $\mathbf{G}\cdot e$, where $\mathfrak{n}$ is an ad_h-invariant complement to $\mathfrak{g}(e)$. We know that the transverse structure on N, equipped with the linear coordinates $q_1, \ldots, q_k$, is a quasi-homogeneous polynomial Poisson structure of generic rank 2. Let $S(\mathfrak{g}^*)^{\mathbf{G}}$ be the algebra of Ad-invariant polynomial functions on $\mathfrak{g}$. By a classical theorem due to Chevalley, $S(\mathfrak{g}^*)^{\mathbf{G}}$ is a polynomial algebra, generated by ℓ homogeneous polynomials $(G_1, \ldots, G_\ell)$, whose degree $d_i := \deg(G_i) = m_i + 1$, where $m_1, \ldots, m_\ell$ are the exponents of $\mathfrak{g}$. These functions are Casimirs of the Lie-Poisson structure on $\mathfrak{g}$.

If we denote by χ_i the restriction of G_i to the transverse slice N, then it follows that these functions are independent Casimirs of the transverse Poisson structure.

2.2. *Simple singularities*

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. The Weyl group $\mathcal{W}$ acts on $\mathfrak{h}$ and the algebra $S(\mathfrak{g}^*)^{\mathbf{G}}$ of Ad-invariant polynomial functions on $\mathfrak{g}$ is isomorphic to $S(\mathfrak{h}^*)^{\mathcal{W}}$, the algebra of $\mathcal{W}$-invariant polynomial functions on $\mathfrak{h}^*$. The inclusion homomorphism $S(\mathfrak{g}^*)^{\mathbf{G}} \hookrightarrow S(\mathfrak{g}^*)$, is dual to a morphism $\mathfrak{g} \to \mathfrak{h}/\mathcal{W}$, called the *adjoint quotient*. Concretely, the adjoint quotient is given by

$$\begin{aligned} G : \mathfrak{g} &\to \mathbf{C}^\ell \\ x &\mapsto (G_1(x),\, G_2(x), \ldots, G_\ell(x)). \end{aligned}$$

The zero-fiber $G^{-1}(0)$ of G is exactly the nilpotent variety $\mathcal{N}$ of $\mathfrak{g}$. We are interested in $N \cap \mathcal{N} = N \cap G^{-1}(0) = \chi^{-1}(0)$, which is an affine surface with an isolated, simple singularity.

Up to conjugacy, there are five types of finite subgroups of $\mathbf{SL}_2 = \mathbf{SL}_2(\mathbf{C})$, the cyclic, dihedral and three exceptional types, denoted by

$\mathcal{C}_p, \mathcal{D}_p, \mathcal{T}, \mathcal{O}$ and $\mathcal{I}$. Given such a subgroup $\mathbf{F}$, one looks at the corresponding ring of invariant polynomials $\mathbf{C}[u,v]^{\mathbf{F}}$. In each of the five cases, $\mathbf{C}[u,v]^{\mathbf{F}}$ is generated by three fundamental polynomials X, Y, Z, subject to only one relation $R(X,Y,Z) = 0$, hence the quotient space $\mathbf{C}^2/\mathbf{F}$ can be identified, as an affine surface, with the singular surface in $\mathbf{C}^3$, defined by $R = 0$. The origin is its only singular point; it is called a *(homogeneous) simple singularity*. The exceptional divisor of the minimal resolution of $\mathbf{C}^2/\mathbf{F}$ is a finite set of projective lines. If two of these lines meet, then they meet in a single point, and transversally. Moreover, the intersection pattern of these lines forms a graph that coincides with one of the simply laced Dynkin diagrams of type A_ℓ, D_ℓ, E_6, E_7 or E_8. This type is called the type of the singularity.

For the other simple Lie algebras (of type B_ℓ, C_ℓ, F_4 or G_2), there exists a similar correspondence. By definition, an *(inhomogeneous) simple singularity* of type Δ is a couple (V, Γ) consisting of a homogeneous simple singularity $V = \mathbf{C}^2/\mathbf{F}$ and a group $\Gamma = \mathbf{F}'/\mathbf{F}$ of automorphisms of V.

We can now state the following extension of a theorem of Brieskorn, which is due to Slodowy[7]

Theorem 2.1. *Let $\mathfrak{g}$ be a simple complex Lie algebra, with Dynkin diagram of type Δ. Let $\mathcal{O}_{sr} = \mathbf{G} \cdot e$ be the subregular orbit and $N = e + \mathfrak{n}^\perp$ a transverse slice to $\mathbf{G} \cdot e$. The surface $N \cap \mathcal{N} = \chi^{-1}(0)$ has a (homogeneous or inhomogeneous) simple singularity of type Δ.*

2.3. *The determinantal Poisson structure*

In terms of linear coordinates $q_1, q_2, \ldots, q_{\ell+2}$ on $\mathbf{C}^{\ell+2}$, the formula

$$\{f, g\}_{det} := \frac{df \wedge dg \wedge d\chi_1 \wedge \cdots \wedge d\chi_\ell}{dq_1 \wedge dq_2 \wedge \cdots \wedge dq_{\ell+2}}$$

defines a Poisson bracket on $\mathbf{C}^{\ell+2}$ with Casimirs $\chi_1, \ldots, \chi_\ell$.

In our case it means that we have two polynomial Poisson structures on the transverse slice N which have $\chi_1, \ldots, \chi_\ell$ as Casimirs on $N \cong \mathbf{C}^{\ell+2}$, namely the transverse Poisson structure and the determinantal structure, constructed by using these Casimirs. The fact that both structures have the same quasi-degree -2, combined with the fact that the singularity of $\chi^{-1}(0)$ is isolated yields the following result :

Theorem 2.2. *Let $\mathcal{O}_{sr}$ be the subregular nilpotent adjoint orbit of a complex semi-simple Lie algebra $\mathfrak{g}$ and let (h, e, f) be the canonical triple, associated to $\mathcal{O}_{sr}$. Let $N = e + \mathfrak{n}^\perp$ be a transverse slice to $\mathcal{O}_{sr}$, where $\mathfrak{n}$ is an* ad_h*-invariant complementary subspace to $\mathfrak{g}(e)$. Let $\{\cdot,\cdot\}_N$ and $\{\cdot,\cdot\}_{det}$*

denote respectively the transverse Poisson structure and the determinantal structure on N. Then $\{\cdot\,,\cdot\}_N = c\,\{\cdot\,,\cdot\}_{det}$ for some $c \in \mathbf{C}^*$.

2.4. *Reduction to a 3×3 Poisson matrix*

Let $\mathcal{O}_{sr}$ be the subregular nilpotent adjoint orbit of a complex semi-simple Lie algebra $\mathfrak{g}$ of rank ℓ.

Our goal now is to show that, in well-chosen coordinates, the transverse Poisson structure $\{\cdot\,,\cdot\}_N$ on N is essentially given by a 3×3 skew-symmetric matrix, closely related to the polynomial that defines the singularity. The non-Poisson part of the following theorem is due to Brieskorn in the case of ADE singularities and was extended later to the other types of simple Lie algebras by Slodowy.[7] Brieskorn's theorem says that the map $\chi : N \to \mathbf{C}^\ell$, which is the restriction of the adjoint quotient to the slice N, is a semi-universal deformation of the singular surface $N \cap \mathcal{N}$. Using these results and the determinantal formula we obtain the following result:

Theorem 2.3. *After possibly relabeling the coordinates q_i and the Casimirs χ_i, the $\ell + 2$ functions*

$$\chi_i, 1 \leq i \leq \ell - 1, \quad \textit{and} \quad q_\ell,\, q_{\ell+1},\, q_{\ell+2}$$

form a system of coordinates on the affine space N. The Poisson matrix of the transverse Poisson structure on N takes, in terms of these coordinates, the form

$$\Lambda_N = \begin{pmatrix} 0 & 0 \\ 0 & \Omega \end{pmatrix} ,$$

where

$$\Omega = \begin{pmatrix} 0 & \dfrac{\partial \chi_\ell}{\partial q_{\ell+2}} & -\dfrac{\partial \chi_\ell}{\partial q_{\ell+1}} \\ -\dfrac{\partial \chi_\ell}{\partial q_{\ell+2}} & 0 & \dfrac{\partial \chi_\ell}{\partial q_\ell} \\ \dfrac{\partial \chi_\ell}{\partial q_{\ell+1}} & -\dfrac{\partial \chi_\ell}{\partial q_\ell} & 0 \end{pmatrix} .$$

It has the polynomial χ_ℓ as Casimir, which reduces to the polynomial which defines the singularity, when setting $\chi_j = 0$ for $j = 1, 2, \ldots, \ell - 1$.

3. The minimal orbit

In this section we consider the transverse Poisson structure to the minimal orbit $\mathcal{O}_m$ in an arbitrary semi-simple Lie algebra $\mathfrak{g}$, whose Killing form will be denoted by $\langle \cdot \mid \cdot \rangle$. This orbit is the nilpotent orbit of minimal dimension (besides the trivial orbit $\{0\}$). It is unique and is generated by a root vector E_m, associated to a highest root, with respect to a fixed Cartan subalgebra $\mathfrak{h}$ and a choice of simple roots. Let (H_m, E_m, F_m) denote the canonical triplet, associated to $\mathcal{O}_m$ and let $\mathfrak{g}^{E_m}$ denote the centralizer of E_m in $\mathfrak{g}$.

3.1. *Properties of the minimal orbit*

We first list the properties of the minimal orbit that we will use (see[2] for proofs). The Lie algebra $\mathfrak{g}$ decomposes in eigenspaces, relatively to ad_{H_m}, as follows:

$$\mathfrak{g} = \mathfrak{g}(-2) \oplus \mathfrak{g}(-1) \oplus \mathfrak{g}(0) \oplus \mathfrak{g}(1) \oplus \mathfrak{g}(2).$$

This decomposition has the following properties.

(F1) $\mathfrak{g}(2) = \mathbf{C}.E_m$;

(F2) $\mathfrak{g}(-2) = \mathbf{C}.F_m$;

(F3) $\mathfrak{g}(0)$ is a reductive subalgebra of $\mathfrak{g}$ and $\mathfrak{g}(0) = \mathfrak{g}^{E_m}(0) \oplus \mathbf{C}.H_m$, where $\mathfrak{g}^{E_m}(0) := \mathfrak{g}^{E_m} \cap \mathfrak{g}(0)$;

(F4) $\mathfrak{g}^{E_m}(0) \perp \mathbf{C}.H_m$;

(F5) $\mathfrak{g}^{E_m}(0)$ is reductive, $\mathfrak{g}^{E_m}(0) = \mathbf{Z}_e \oplus \mathfrak{m}_e$ where $\mathbf{Z}_e \subset \mathfrak{h}$ is the center of $\mathfrak{g}^{E_m}(0)$ and $\mathfrak{m}_e$ its semi-simple part;

(F6) Let $\mathfrak{n}_e^+ := \mathfrak{m}_e \cap \mathfrak{n}^+ = \langle X_\alpha,\, \alpha(H_m) = 0 \rangle$, let $\mathfrak{n}_e^-$ denote its opposite and let $\mathfrak{h}_e := \mathfrak{m}_e \cap \mathfrak{h}$. Then $\mathfrak{m}_e = \mathfrak{n}_e^- \oplus \mathfrak{h}_e \oplus \mathfrak{n}_e^+$ and $\mathfrak{h} = \mathfrak{h}_e \oplus \mathbf{Z}_e \oplus \mathbf{C}.H_m$.

(F7) Let K_f be the skew-symmetric bilinear form defined by $K_f(X,Y) := \langle F_m | [X,Y] \rangle$. The space $\mathfrak{g}(1)$, equipped with K_f, is a symplectic space of dimension $2s$. Moreover, we can choose a basis $(Z_1, \dots, Z_s, Z_{s+1}, \dots, Z_{2s})$ such that each vector Z_i is a root vector X_{α_i}, associated to a positive root, and

$$[Z_i, Z_j] = 0 = [Z_{i+s}, Z_{j+s}], \qquad [Z_i, Z_{2s+1-j}] = \delta_{ij} E_m,$$

for all i, j with $1 \le i, j \le s$;

(F8) The same result as in (F7) holds for the space $\mathfrak{g}(-1)$ equipped with the bilinear form $K_e(X,Y) = \langle E_m | [X,Y] \rangle$. The corresponding basis, defined by the same properties as in (F7), will be denoted by $X_1, \dots, X_{2s}$.

3.2. *The ATP-structure associated to $\mathcal{O}_m$*

Since the centralizer $\mathfrak{g}^{E_m}$ is given by

$$\mathfrak{g}^{E_m} = \mathfrak{g}^{E_m}(0) \oplus \mathfrak{g}(1) \oplus \mathfrak{g}(2),$$

we have a decomposition $\mathfrak{g} = \mathfrak{g}^{E_m} \oplus \mathfrak{n}$, where $\mathfrak{n} = \mathfrak{g}(-2) \oplus \mathfrak{g}(-1) \oplus \mathbf{C}.H_m$. It is clear that the Lie subalgebra $\mathfrak{n}$ of $\mathfrak{g}$ is ad_{H_m}-invariant, has dimension $2s+2$ and that its orthogonal is given by $\mathfrak{n}^\perp = \mathfrak{g}(-2) \oplus \mathfrak{g}(-1) \oplus \mathfrak{g}^{E_m}(0)$. Thus, we choose $N_m := E_m + \mathfrak{n}^\perp$ as transverse slice to $\mathcal{O}_m$. The Poisson matrix of the corresponding ATP-Poisson structure on N_m is given, at $n \in N_m$, by the Dirac formula

$$\Lambda_m(n) = A(n) + B(n)C^{-1}(n)B(n)^T,$$

where the matrices $A(n), B(n), C(n)$ are constructed as follows. Let $Z_{2s+2}, \ldots, Z_{2s+p+1}$ be a basis of $\mathfrak{g}^{E_m}(0)$ and let $Z_{2s+1} := E_m$, so that $Z_1, \ldots, Z_{2s+p+1}$ is a basis of $\mathfrak{g}^{E_m}$. Also, set $X_{2s+1} := \frac{1}{2}H_m$ and $X_{2s+2} := F_m$. Then, $X_1, \ldots, X_{2s+2}$ is a basis of $\mathfrak{n}$. In terms of these bases, the matrices $A(n), B(n)$ and $C(n)$ are given by

$$A(n)_{ij} = \langle n | [Z_i, Z_j] \rangle, \quad B(n)_{ik} = \langle n | [Z_i, X_k] \rangle, \quad C(n)_{kl} = \langle n | [X_k, X_l] \rangle,$$

where $1 \le i, j \le 2s+p+1$ and $1 \le k, l \le 2s+2$. In terms of the $2q \times 2q$-matrices J_q, defined by

$$J_q = \begin{pmatrix} 0 & & \ldots & & 0 & 1 \\ & & & & \iddots & 0 \\ \vdots & & \iddots & 1 & & \\ & & -1 & \iddots & & \vdots \\ 0 & \iddots & & & & \\ -1 & 0 & & \ldots & & 0 \end{pmatrix},$$

we have:

$$C(n) = \begin{pmatrix} J_s & 0 \\ 0 & -J_2 \end{pmatrix} \qquad \text{and} \qquad C^{-1}(n) = -C(n).$$

Proposition 3.1. *The ATP-structure of the minimal orbit $\mathcal{O}_m$ is the sum of two Poisson structures $\Lambda_m = \mathcal{A} + \mathcal{Q}$, where*

1. *$\mathcal{A}$ is a linear Poisson structure, isomorphic to the Lie-Poisson structure on the dual of the Lie algebra $\mathfrak{g}^{E_m}$;*
2. *$\mathcal{Q}$ is a quadratic Poisson bracket, whose Poisson matrix at $n \in N_m$ is given by $Q(n) := B(n)C^{-1}(n)B(n)^T$. Moreover, its generic rank is $\dim \mathcal{O}_m - 2$.*

Proof. The first statement is clear because the matrix $A(n)$ in the Dirac formula is the matrix of the Lie-Poisson structure on $(\mathfrak{g}^{E_m})^*$. Both matrices A and $\Lambda_m = A+Q$ are Poisson. Moreover, since $C(n)$ is constant (independent of $n \in N_m$), all entries of Q are quadratic polynomials. Since Q is the highest degree term of the Poisson matrix Λ, it is also a Poisson matrix. Therefore $\mathcal{A}$ and $\mathcal{Q}$ are compatible Poisson structures. We show that the rank of $\mathcal{Q}$ is $\dim \mathcal{O}_m - 2$. To do this let n be any element in N_m. We will restrict our attention now to the matrix $B(n)$. From the definitions, we get

$$[Z_i, F_m] = [Z_i, H_m] = 0, \qquad [Z_i, X_k] \in \mathfrak{g}(-1),$$

for i, k such that $2s+2 \leq i \leq 2s+p+1$ and $1 \leq k \leq 2s$. It implies that $B(n)_{ik} = 0$ for the latter values of i and k. Thus, the last p rows of $B(n)$ are zero,

$$B(n) = \begin{pmatrix} D(n) \\ 0 \end{pmatrix},$$

where $D(n)$ is the $(2s+1) \times (2s+2)$-matrix, whose entries are given by

$$D(n)_{ik} = \langle n \mid [Z_i, X_k] \rangle,$$

where $1 \leq i \leq 2s+1$ and $1 \leq k \leq 2s+2$. If $1 \leq i \leq 2s$ then $[Z_i, X_{2s+2}] \in \mathfrak{g}(-1)$ and $[E_m, X_{2s+2}] = H_m$. Using (F4), it follows that the last column of the matrix $D(n)$ is zero,

$$D(n) = \begin{pmatrix} D'(n) & 0 \end{pmatrix}.$$

Thus, for $n \in E_m + \mathfrak{n}^\perp$, we have

$$Q(n) = \begin{pmatrix} D(n)C^{-1}(n)D(n)^T & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} D'(n)C'(n)D'(n)^T & 0 \\ 0 & 0 \end{pmatrix},$$

where $C'(n)$ is the submatrix of $C^{-1}(n)$, obtained by removing its last column and its last row. This implies that

$$\mathrm{Rk}(Q(n)) = \mathrm{Rk}(D'(n)C'(n)D'(n)^T) \leq 2s+1,$$

for all $n \in E_m + \mathfrak{n}^\perp$. Since Q is skew-symmetric, its rank is at most $2s = \dim \mathcal{O}_m - 2$. We show that there exists a point $n \in N_m$ where the rank of Q is $2s$. Recall that the symplectic vector space $\mathfrak{g}(1)$ is generated by root vectors, $Z_i = X_{\alpha_i}$, for $1 \leq i \leq 2s$. So, we can define $P_i := \mathfrak{n}^\perp \cap H_{\alpha_i}^\perp$, for $1 \leq i \leq 2s$, which are hyperplanes of $\mathfrak{n}^\perp$. Let P denote their union. Let $n \in \mathfrak{n}^\perp \backslash P + E_m$. Then

1. If $1 \leq i \neq k \leq 2s$ then $[Z_i, X_k] \subset \mathfrak{n}_e^- \oplus \mathfrak{n}_e^+ \subset V$, so $D'(n)_{ik} = \langle n \mid [Z_i, X_k] \rangle = 0$;

2. For all i with $1 \leq i \leq 2s$, $D'(n)_{ii} = \langle n \mid H_{\alpha_i} \rangle \neq 0$;
3. For all k with $1 \leq k \leq 2s+1$, $[Z_{2s+1}, X_k] \in \mathfrak{g}(1) \oplus \mathfrak{g}(2)$, so that $D'(n)_{2s+1,k} = 0$.

Consider the submatrix C'' of C', and similarly D'' of D', obtained by removing its last row and its last column. Then the upper left $2s \times 2s$-minor of $D'(n)C'(n)D'(n)^T$ is $\det\left(D''(n)C''(n)D''(n)^T\right)$, which is non-zero, This proves that $\mathrm{Rk}(Q(n)) = 2s = \dim \mathcal{O}_m - 2$, so that $\mathrm{Rk}(\mathcal{Q}) = \dim \mathcal{O}_m - 2$.$\square$

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