TRANSVERSE POISSON STRUCTURES TO ADJOINT ORBITS IN SEMI-SIMPLE LIE ALGEBRAS

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Abstract. We study the transverse Poisson structure to adjoint orbits in a complex semi-simple Lie algebra. The problem is first reduced to the case of nilpotent orbits. We prove then that in suitably chosen quasi-homogeneous coordinates the quasi-degree of the transverse Poisson structure is -2. In the particular case of subregular nilpotent orbits we show that the structure may be computed by means of a simple determinantal formula, involving the restriction of the Chevalley invariants on the slice. In addition, using results of Brieskorn and Slodowy, the Poisson structure is reduced to a three dimensional Poisson bracket, intimately related to the simple rational singularity that corresponds to the subregular orbit.

1. Introduction

The transverse Poisson structure was introduced by A. Weinstein in [16], stating in his famous splitting theorem that every (real smooth or complex holomorphic) Poisson manifold M is, in the neighbourhood of each point m, the product of a symplectic manifold and a Poisson manifold of rank 0 at m. The two factors of this product can be geometrically realized as follows: let S be the symplectic leaf through m and let N be any submanifold of M containing m such that

$$T_m(M) = T_m(S) \oplus T_m(N).$$

There exists a neighbourhood V of m in N, endowed with a Poisson structure, and a neighbourhood U of m in S such that, near m, M is isomorphic to the product Poisson manifold $U \times V$. The submanifold N is called *a transverse slice* at m to the symplectic leaf S and the Poisson structure on $V \subset N$ is called *the transverse Poisson structure* to S. Up to Poisson isomorphism, it is independent of the point $m \in S$ and of the chosen transverse slice N at m: given two points m, $m' \in S$ and two transverse slices N, N' at m resp. m' to S, there exist neighbourhoods V of m in N and V' of m' in N' such that (V, m) and (V', m') are Poisson diffeomorphic.

When M is the dual $\mathfrak{g}^*$ of a complex Lie algebra $\mathfrak{g}$, equipped with its standard Lie-Poisson structure, we know that the symplectic leaf through $\mu \in \mathfrak{g}^*$ is the co-adjoint orbit $\mathbf{G} \cdot \mu$ of the adjoint Lie group $\mathbf{G}$ of $\mathfrak{g}$. In this case, a natural transverse slice to $\mathbf{G} \cdot \mu$ is obtained in the following way: we choose any complement $\mathfrak{n}$ to the centralizer $\mathfrak{g}(\mu)$ of μ in $\mathfrak{g}$ and we take N to be the affine subspace $\mu + \mathfrak{n}^\perp$ of $\mathfrak{g}^*$. Since $\mathfrak{g}(\mu)^\perp = \mathrm{ad}^*_{\mathfrak{g}}\,\mu$ we have

$$T_\mu(\mathfrak{g}^*) = T_\mu(\mathbf{G} \cdot \mu) \oplus T_\mu(N),$$

so that N is indeed a transverse slice to $\mathbf{G} \cdot \mu$ at μ. Furthermore, defining on $\mathfrak{n}^\perp$ any system of linear coordinates $(q_1, \dots, q_k)$, and using the explicit formula for Dirac reduction

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(see Formula (4) below), one can write down explicit formulas for the Poisson matrix $\Lambda_N := \left(\{q_i, q_j\}_N\right)_{1\leq i,j\leq k}$ of the transverse Poisson structure, from which it follows easily that the coefficients of Λ_N are actually rational functions in $(q_1, \dots, q_k)$. As a corollary, in the Lie-Poisson case, the transverse Poisson structure is always rational ([11]). One immediately wonders in which cases the Poisson structure on N is polynomial; more precisely, for which Lie algebras $\mathfrak{g}$, for which co-adjoint orbits for which complement $\mathfrak{n}$.

Partial answers have been given in the literature for (co-) adjoint orbits in a semi-simple Lie algebra. In ([7]), P. Damianou computed explicitly the transverse Poisson structure to nilpotent orbits of $\mathfrak{gl}_n$, for $n \leq 7$, corresponding to a particular complement $\mathfrak{n}$, yielding that in this case the transverse Poisson structure is polynomial. In ([6]), R. Cushman and M. Roberts proved that there exists for any nilpotent adjoint orbit of a semi-simple Lie algebra a special choice of a complement $\mathfrak{n}$ such that the corresponding transverse Poisson structure is polynomial. For the latter case, H. Sabourin gave in ([10]) a more general class of complements for which the transverse structure is polynomial, using in an essential way the machinery of semi-simple Lie algebras; he also showed that the choice of complement $\mathfrak{n}$ is relevant for the polynomial character of the transverse Poisson structure by giving an example in which where the latter structure is rational for a generic choice of complement.

When the transverse Poisson structure is polynomial one is tempted to define its degree as the maximal degree of the coefficients $\{q_i, q_j\}_N$ of its Poisson matrix, as was done in [7] and [6], who also formulate several conjectures about this degree. Unfortunately, as shown in [10], this degree depends strongly on the choice of the complement $\mathfrak{n}$, hence it is not intrinsically attached to the transverse Poisson structure. We show in Section 3 that the right approach is by using the more general notion of "quasi-degree", i.e., we assign natural quasi-degrees $\varpi(q_i)$ to the variables q_i $(i = 1, \dots, k)$ and we show that, in the above mentioned class of complements, the quasi-degree of the transverse Poisson structure is always -2, irrespective of the simple Lie algebra, the chosen adjoint orbit and the chosen transverse slice N! In fact, the weights $\varpi(q_i)$ have a Lie-theoretic origin and are also independent of the particular complement. It follows that $\{q_i, q_j\}_N$ is a quasi-homogeneous polynomial of quasi-degree $\varpi(q_i) + \varpi(q_j) - 2$, for $1 \leq i, j \leq k$.

Another result, established in this article, is that the study of the transverse Poisson structure to any adjoint orbit $\mathbf{G} \cdot x$ can be reduced, via the Jordan decomposition of $x \in \mathfrak{g}$, to the case of an adjoint *nilpotent* orbit. Thereby we explain why we are merely interested in the case of nilpotent orbits.

It is easy to observe that the transverse structure to the regular nilpotent orbit $\mathcal{O}_{reg}$ of $\mathfrak{g}$ is always trivial. So, the next step is to consider the case of the *subregular* nilpotent orbit $\mathcal{O}_{sr}$ of $\mathfrak{g}$. Then $N \cong \mathbf{C}^{\ell+2}$, where ℓ is the rank of $\mathfrak{g}$. The dimension of $\mathcal{O}_{sr}$ is two less than the dimension of the regular orbit, so that the transverse Poisson structure has rank 2. It has ℓ independent polynomial Casimirs functions $\chi_1, \dots, \chi_\ell$, where χ_i is the restriction of the i-th Chevalley invariant G_i to the slice N. In this case the transverse Poisson structure may be obtained by a simple determinantal formula instead of the usual rather complicated Dirac's constraint formula. The determinantal formula may be formulated as follows: in terms of linear coordinates $q_1, q_2, \dots, q_{\ell+2}$ on N, the formula

$$\{f, g\}_{det} := \frac{df \wedge dg \wedge d\chi_1 \wedge \cdots \wedge d\chi_\ell}{dq_1 \wedge dq_2 \wedge \ldots \wedge dq_{\ell+2}} \tag{1}$$

defines a Poisson bracket on N, which coincides (up to a non-zero constant), with the transverse Poisson structure on N.

As an application of Formula (1), we show in Theorem 5.5 that the Poisson matrix of the transverse Poisson on N takes, in suitable coordinates, the block form

$$\widetilde{\Lambda}_N = \begin{pmatrix} 0 & 0 \\ 0 & \Omega \end{pmatrix}, \quad \text{where} \quad \Omega = \begin{pmatrix} 0 & \dfrac{\partial F}{\partial q_{\ell+2}} & -\dfrac{\partial F}{\partial q_{\ell+1}} \\ -\dfrac{\partial F}{\partial q_{\ell+2}} & 0 & \dfrac{\partial F}{\partial q_{\ell}} \\ \dfrac{\partial F}{\partial q_{\ell+1}} & -\dfrac{\partial F}{\partial q_{\ell}} & 0 \end{pmatrix}.$$

The polynomial $F = F(u_1, \dots, u_{\ell-1}, q_\ell, q_{\ell+1}, q_{\ell+2})$ that appears in this formula is precisely the polynomial that describes the universal deformation of the (homogeneous or inhomogeneous) simple singularity of the singular surface $N \cap \mathcal{N}$, where $u_1, \dots, u_{\ell-1}$ are the deformation parameters, which are Casimirs for the Poisson structure on N, and where $\mathcal{N}$ is the nilpotent cone of $\mathfrak{g}$. In particular, the restriction of this Poisson structure to $N \cap \mathcal{N}$, is given by

$$\{x, y\} = \frac{\partial F_0}{\partial z}, \qquad \{y, z\} = \frac{\partial F_0}{\partial x}, \qquad \{z, x\} = \frac{\partial F_0}{\partial y},$$

where $F_0(x, y, z) := F(0, \dots, 0, x, y, z)$ is the polynomial that defines $N \cap \mathcal{N}$, as a surface in $\mathbf{C}^3$. As we will recall in Paragraph 5.4, Brieskorn [4] showed that, in the ADE case, the so-called adjoint quotient, $G = (G_1, \dots, G_\ell) : \mathfrak{g} \to \mathbf{C}^\ell$, restricted to the slice N, is a semi-universal deformation of the singular surface $N \cap \mathcal{N}$; this result was generalized by Slodowy [12] to the other simple Lie algebras. Our Theorem 5.5 adds a Poisson dimension to this result.

The article is organized as follows: In Section 2, we recall a few basic facts concerning transverse Poisson structures and we show that the case of a general orbit in a semi-simple Lie algebra can be reduced to the case of a nilpotent orbit. In Section 3, we recall the notion of quasi-homogeneity and we show that, for a natural class of slices, the transverse Poisson structure is quasi-homogeneous of quasi-degree -2. In Section 4 and the end of Section 5, we show in some examples, namely the Lie algebras $\mathfrak{g}_2$, $\mathfrak{so}_8$ and $\mathfrak{sl}_4$, how the transverse Poisson structure can be computed explicitly, and we use these examples to illustrate our results. In Section 5 we prove that in the case of the subregular orbit, the transverse Poisson structure is given by a determinantal formula; we also show in that section that this Poisson structure is entirely determined by the singular variety of nilpotent elements of the slice.

2. Transverse Poisson structures in semi-simple Lie algebras

In this section we recall the main setup for studying the transverse Poisson structure to a (co-) adjoint orbit in the case of a complex semi-simple Lie algebra $\mathfrak{g}$, and we show how the case of a general orbit is related to the case of a nilpotent orbit. We use the Killing form $\langle \cdot \mid \cdot \rangle$ of $\mathfrak{g}$ to identify $\mathfrak{g}$ with its dual $\mathfrak{g}^*$, which leads to a Poisson structure on $\mathfrak{g}$, given for functions F, G on $\mathfrak{g}$ at $x \in \mathfrak{g}$ by

$$\{F, G\}(x) := \langle x \mid [dF(x), dG(x)] \rangle, \tag{2}$$

where we think of $dF(x)$ and $dG(x)$ as elements of $\mathfrak{g} \cong \mathfrak{g}^* \cong T_x^*\mathfrak{g}$. Since the Killing form is Ad-invariant, the isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$ identifies the adjoint orbits $\mathbf{G} \cdot x$ of $\mathbf{G}$ with the co-adjoint orbits $\mathbf{G} \cdot \mu$, so the symplectic leaf of $\{\cdot\,, \cdot\}$ that passes through x is the adjoint orbit $\mathbf{G} \cdot x$. Also, as a transverse slice at x to $\mathbf{G} \cdot x$ we can take an affine subspace $N := x + \mathfrak{n}^\perp$, where $\mathfrak{n}$ is any complementary subspace to the centralizer

$\mathfrak{g}(x) := \{y \in \mathfrak{g} \mid [x, y] = 0\}$ of x in $\mathfrak{g}$ and $\perp$ is the orthogonal complement with respect to the Killing form. In order to give an explicit formula for the transverse Poisson structure $\{\cdot\,,\cdot\}_N$ to $\mathbf{G} \cdot x$, let $(Z_1, \dots, Z_k)$ be a basis for $\mathfrak{g}(x)$ and let $(X_1, \dots, X_{2r})$ be a basis for $\mathfrak{n}$, where $2r = \dim(\mathbf{G} \cdot x)$ is the rank of the Poisson structure (2) at x. These bases lead to linear coordinates $q_1, \dots, q_{k+2r}$ on $\mathfrak{g}$, centered at x, defined by $q_i(y) := \langle y - x \,|\, Z_i\rangle$, for $i = 1, \dots, k$ and $q_{k+i}(y) := \langle y - x \,|\, X_i\rangle$, for $i = 1, \dots, 2r$. Since $dq_i(y) = Z_i$ for $i = 1, \dots, k$ and $dq_{k+i}(y) = X_i$ for $i = 1, \dots, 2r$, it follows from (2) that the Poisson matrix of $\{\cdot\,,\cdot\}$ at $y \in \mathfrak{g}$ is given by

$$(\{q_i, q_j\}(y))_{1\le i,j\le k+2r} = \begin{pmatrix} A(y) & B(y) \\ -B(y)^\top & C(y) \end{pmatrix}, \tag{3}$$

where

$$\begin{aligned} A_{i,j}(y) &= \langle y \,|\, [Z_i, Z_j]\rangle && 1 \le i, j \le k; \\ B_{i,m}(y) &= \langle y \,|\, [Z_i, X_m]\rangle && 1 \le i \le k, \quad 1 \le m \le 2r; \\ C_{l,m}(y) &= \langle y \,|\, [X_l, X_m]\rangle && 1 \le l, m \le 2r. \end{aligned}$$

It is easy to see that the skew-symmetric matrix $C(x)$ is invertible, so $C(y)$ is invertible for y in a neighborhood of x in $\mathfrak{g}$, and hence for y in a neighborhood V of x in N. By Dirac reduction, the Poisson matrix of $\{\cdot\,,\cdot\}_N$ at $n \in V$, in terms of the coordinates $q_1, \dots, q_k$ (restricted to V), is given by

$$\Lambda_N(n) = A(n) + B(n)C(n)^{-1}B(n)^\top. \tag{4}$$

According to the Jordan-Chevalley decomposition theorem we can write x as $x = s + e$, where s is semi-simple, e is nilpotent and $[s, e] = 0$. Moreover, the respective centralizers of x, s and e are related as follows:

$$\mathfrak{g}(x) = \mathfrak{g}(s) \cap \mathfrak{g}(e). \tag{5}$$

This leads to a natural class of complements $\mathfrak{n}$ to $\mathfrak{g}(x)$; Since the restriction of $\langle\cdot \,|\, \cdot\rangle$ to $\mathfrak{g}(s)$ is non-degenerate (see [8, Prop. 1.7.7.]), we have a vector space decomposition of $\mathfrak{g}$ as

$$\mathfrak{g} = \mathfrak{g}(s) \oplus \mathfrak{n}_s,$$

where $\mathfrak{n}_s = \mathfrak{g}(s)^\perp$. Notice that $\mathfrak{n}_s$ is $\mathfrak{g}(s)$-invariant, $[\mathfrak{g}(s), \mathfrak{n}_s] \subset \mathfrak{n}_s$, since

$$\langle \mathfrak{g}(s) \,|\, [\mathfrak{g}(s), \mathfrak{n}_s]\rangle = \langle [\mathfrak{g}(s), \mathfrak{g}(s)] \,|\, \mathfrak{n}_s\rangle \subset \langle \mathfrak{g}(s) \,|\, \mathfrak{n}_s\rangle = \{0\}\,.$$

Choosing any complement $\mathfrak{n}_e$ of $\mathfrak{g}(x)$ in $\mathfrak{g}(s)$ we get the following decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{g}(x) \oplus \mathfrak{n}_e \oplus \mathfrak{n}_s.$$

We take then $\mathfrak{n} := \mathfrak{n}_e \oplus \mathfrak{n}_s$ and we denote $N_x := x + \mathfrak{n}^\perp$. It follows that, if $n \in N_x$, so that $n \in \mathfrak{g}(s)$, then $\langle n \,|\, [\mathfrak{g}(s), \mathfrak{n}_s]\rangle \subset \langle \mathfrak{g}(s) \,|\, \mathfrak{n}_s\rangle = \{0\}$, so that in particular

$$\langle n \,|\, [\mathfrak{g}(x), \mathfrak{n}_s]\rangle = \{0\} \quad \text{and} \quad \langle n \,|\, [\mathfrak{n}_e, \mathfrak{n}_s]\rangle = \{0\}\,. \tag{6}$$

Let us assume that the basis vectors $X_1, \dots, X_{2r}$ of $\mathfrak{n}$ have been chosen such that $X_1, \dots, X_{2p} \in \mathfrak{n}_e$ and $X_{2p+1}, \dots, X_{2r} \in \mathfrak{n}_s$. Then the formulas (6) imply that the Poisson matrix (3) takes at $n \in N_x$ the form

$$\Lambda(n) = \begin{pmatrix} A(n) & B_e(n) & 0 \\ -B_e(n)^\top & C_e(n) & 0 \\ 0 & 0 & C_s(n) \end{pmatrix},$$

where

$$\begin{aligned}
A_{i,j}(n) &= \langle n \,|\, [Z_i, Z_j] \rangle && 1 \le i, j \le k; \\
B_{e;i,m}(n) &= \langle n \,|\, [Z_i, X_m] \rangle && 1 \le i \le k, \quad 1 \le m \le 2p; \\
C_{e;l,m}(n) &= \langle n \,|\, [X_l, X_m] \rangle && 1 \le l, m \le 2p; \\
C_{s;l,m}(n) &= \langle n \,|\, [X_l, X_m] \rangle && 2p+1 < l, m \le 2r.
\end{aligned}$$

It follows from (4) that the Poisson matrix of the transverse Poisson structure on N_x is given by

$$\Lambda_{N_x}(n) = A(n) + B_e(n) C_e(n)^{-1} B_e(n)^\top. \tag{7}$$

Let us now restrict our attention to the Lie algebra $\mathfrak{g}(s)$, which is reductive, so it decomposes as

$$\mathfrak{g}(s) = \mathfrak{z}(s) \oplus \mathfrak{g}_{ss}(s),$$

where $\mathfrak{z}(s)$ is the center of $\mathfrak{g}(s)$ and $\mathfrak{g}_{ss}(s) = [\mathfrak{g}(s), \mathfrak{g}(s)]$ is the semi-simple part of $\mathfrak{g}(s)$. At the group level we have a similar decomposition of $\mathbf{G}(s)$, the centralizer of s in $\mathbf{G}$, whose Lie algebra is $\mathfrak{g}(s)$, namely

$$\mathbf{G}(s) = \mathbf{Z}(s)\mathbf{G}_{ss}(s),$$

where $\mathbf{Z}(s)$ is a central subgroup of $\mathbf{G}(s)$ and $\mathbf{G}_{ss}(s)$ is the semi-simple part of $\mathbf{G}(s)$, with Lie algebra $\mathfrak{g}_{ss}(s)$. Since $e \in \mathfrak{g}(s)$ we can consider $\mathbf{G}(s) \cdot e$ as an adjoint orbit of the reductive Lie algebra $\mathfrak{g}(s)$. We may think of it as an adjoint orbit of a *semi-simple* Lie algebra, since $\mathbf{G}(s) \cdot e = \mathbf{G}_{ss}(s) \cdot e$; similarly we may think of a transverse slice to the adjoint orbit $\mathbf{G}(s) \cdot e$ as a transverse slice to $\mathbf{G}_{ss}(s) \cdot e$, up to a summand with trivial Lie bracket. Denoting by $\perp_s$ the orthogonal complement with respect to $\langle \cdot \,|\, \cdot \rangle$, restricted to $\mathfrak{g}(s)$, we have that $N := e + \mathfrak{n}_e^{\perp_s}$ is a transverse slice to $\mathbf{G}(s) \cdot e$ since

$$\mathfrak{g}(s) = \mathfrak{g}(x) \oplus \mathfrak{n}_e = \mathfrak{z}(s) \oplus \mathfrak{g}_{ss}(s)(e) \oplus \mathfrak{n}_e.$$

We have used that $\mathfrak{g}(x) = \mathfrak{g}(s)(e)$, the centralizer of e in $\mathfrak{g}(s)$, which follows from (5). In terms of the bases $(Z_1, \dots, Z_k)$ of $\mathfrak{g}(x)$ and $(X_1, \dots, X_{2p})$ of $\mathfrak{n}_e$ that we had picked, the Poisson matrix takes at $n \in N$ the form

$$\begin{pmatrix} A(n) & B_e(n) \\ -B_e(n)^\top & C_e(n) \end{pmatrix},$$

which leads by Dirac reduction to the following formula for the transverse Poisson structure Λ_N on N,

$$\Lambda_N(n) = A(n) + B_e(n) C_e(n)^{-1} B_e(n)^\top,$$

where $n \in N$. This yields formally the same formula as (7), except that it is evaluated at points n of N, rather than at points of N_x. However, since $\mathfrak{n}_e^{\perp_s} = \mathfrak{g}(s) \cap \mathfrak{n}_e^\perp = \mathfrak{n}_s^\perp \cap \mathfrak{n}_e^\perp = (\mathfrak{n}_s + \mathfrak{n}_e)^\perp = \mathfrak{n}^\perp$, the affine subspaces N_x and N only differ by a translation, $N_x = s + e + \mathfrak{n}^\perp = s + N$, so that they, and their Poisson matrices with respect to the coordinates $q_1, \dots, q_k$, can be identified. It leads to the following proposition.

Proposition 2.1. *Let $x \in \mathfrak{g}$ be any element, $\mathbf{G} \cdot x$ its adjoint orbit and $x = s + e$ its Jordan-Chevalley decomposition. Given any complement $\mathfrak{n}_e$ of $\mathfrak{g}(x)$ in $\mathfrak{g}(s)$ and putting $\mathfrak{n} := \mathfrak{n}_s \oplus \mathfrak{n}_e$, where $\mathfrak{n}_s = \mathfrak{g}(s)^\perp$, the parallel affine spaces $N_x := x + \mathfrak{n}^\perp$ and $N := e + \mathfrak{n}^\perp$ are respectively transverse slices to the adjoint orbit $\mathbf{G} \cdot x$ in $\mathfrak{g}$ and to the nilpotent orbit $\mathbf{G}(s) \cdot e$ in $\mathfrak{g}(s)$. The Poisson structure on both transverse slices is given by the same Poisson matrix, namely that of (7), in terms of the same affine coordinates restricted to the corresponding transverse slice.*

In short, the transverse Poisson structure to any adjoint orbit $\mathbf{G}\cdot x$ of a semi-simple (or reductive) algebra $\mathfrak{g}$ is essentially determined by the transverse Poisson structure to the underlying nilpotent orbit $\mathbf{G}(s)\cdot e$ defined by the Jordan decomposition $x = s + e$. A refinement of this proposition will be given in Corollary 3.4.

3. The polynomial and the quasi-homogeneous character of the tranverse Poisson structure

In this section we show that, for a natural class of transverse slices to a nilpotent orbit $\mathcal{O}$, which we equip with an adapted set of linear coordinates, centered at a nilpotent element $e \in \mathcal{O}$, the transverse Poisson structure is quasi-homogeneous (of quasi-degree -2), in the following sense.

Definition 3.1. Let $\nu = (\nu_1, \ldots, \nu_d)$ be non-negative integers. A polynomial $P \in \mathbf{C}[x_1, \ldots, x_d]$ is said to be *quasi-homogeneous* (relative to ν) if for some integer κ,

$$\forall t \in \mathbf{C}, P(t^{\nu_1}x_1, \ldots, t^{\nu_d}x_d) = t^\kappa P(x_1, \ldots, x_d).$$

The integer κ is then called the *quasi-degree* (relative to ν) of P, denoted $\varpi(P)$. Similarly, a polynomial Poisson structure $\{\cdot\,,\cdot\}$ on $\mathbf{C}[x_1, \ldots, x_d]$ is said to be *quasi-homogeneous* (relative to ν) if there exists $\kappa \in \mathbf{Z}$ such that, for any quasi-homogeneous polynomials F and G, their Poisson bracket $\{F, G\}$ is quasi-homogeneous of degree

$$\varpi(\{F, G\}) = \varpi(F) + \varpi(G) + \kappa;$$

equivalently, for any i, j the polynomial $\{x_i, x_j\}$ is quasi-homogeneous of quasi-degree $\nu_i + \nu_j + \kappa$. Then κ is called the *quasi-degree* of $\{\cdot\,,\cdot\}$.

We first show that, given $\mathcal{O}$, we can choose a system of linear coordinates on $\mathfrak{g}$, centered at some nilpotent element $e \in \mathcal{O}$, such that the Lie-Poisson structure on $\mathfrak{g}$ is quasi-homogeneous relative to some vector ν that has a natural Lie-theoretic interpretation. In order to describe how this happens we need to recall a few facts on the theory of semi-simple Lie algebras that will be used throughout this paper. First, one chooses a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, with corresponding root system $\Delta(\mathfrak{h})$, from which a basis $\Pi(\mathfrak{h})$ of simple roots is selected. The *rank* of $\mathfrak{g}$, which is the dimension of $\mathfrak{h}$, is denoted by ℓ. According to the Jacobson-Morosov-Kostant correspondence (see [15, pars. 32.1 and 32.4]), there is a canonical triple (h, e, f) of elements of $\mathfrak{g}$, associated with $\mathcal{O}$ and completely determined by the following properties:

(1) (h, e, f) is a $\mathfrak{sl}_2$-triple, i.e., $[h, e] = 2e$, $[h, f] = -2f$ and $[e, f] = h$;
(2) h is the characteristic of $\mathcal{O}$, i.e., $h \in \mathfrak{h}$ and $\alpha(h) \in \{0, 1, 2\}$ for any simple root $\alpha \in \Pi(\mathfrak{h})$.
(3) $\mathcal{O} = \mathbf{G}\cdot e$.

The triple (h, e, f) leads to two decompositions of $\mathfrak{g}$:

(1) A decomposition of $\mathfrak{g}$ into eigenspaces relative to ad_h. Each eigenvalue being an integer we have

$$\mathfrak{g} = \bigoplus_{i\in\mathbf{Z}} \mathfrak{g}(i),$$

where $\mathfrak{g}(i)$ is the eigenspace of ad_h that corresponds to the eigenvalue i. For example, $e \in \mathfrak{g}(2)$ and $f \in \mathfrak{g}(-2)$.

(2) Let $\mathfrak{s}$ be the Lie subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2$, which is generated by h, e and f. The Lie algebra $\mathfrak{g}$ is an $\mathfrak{s}$-module, hence it decomposes as

$$\mathfrak{g} = \bigoplus_{j=1}^{k} V_{n_j},$$

where each V_{n_j} is a simple $\mathfrak{s}$-module, with $n_j + 1 = \dim V_{n_j}$ ad_h-weights $n_j, n_j - 2, n_j - 4, \ldots, -n_j$. Moreover, $k = \dim \mathfrak{g}(e)$, since the centralizer $\mathfrak{g}(e)$ is generated by the highest weight vectors of each V_{n_j}. It follows that

$$\sum_{j=1}^{k} n_j = \dim \mathfrak{g} - k = \dim(\mathbf{G} \cdot e) = 2r. \tag{8}$$

We pick a system of linear coordinates on $\mathfrak{g}$, centered at e, by using Slodowy's action. We recall the construction of this action from [13]. First, he considers the one-parameter subgroup of $\mathbf{G}$,

$$\begin{array}{rcccl} \lambda & : & \mathbf{C}^* & \to & \mathbf{G} \\ & & t & \mapsto & \exp(\lambda_t h) \end{array}$$

where λ_t is a complex number such that $e^{-\lambda_t} = t$. The restriction of Ad to this subgroup leaves every eigenspace $\mathfrak{g}(i)$ invariant, and acts for each t as a homothecy with ratio t^{-i} on $\mathfrak{g}(i)$:

$$\forall x \in \mathfrak{g}(i), \ \mathrm{Ad}_{\lambda(t)}\, x = t^{-i} x. \tag{9}$$

Since $e \in \mathfrak{g}(2)$ the action ρ of $\mathbf{C}^*$ on $\mathfrak{g}$, defined for $t \in \mathbf{C}^*$ and for $y \in \mathfrak{g}$ by $\rho_t \cdot y := t^2 \,\mathrm{Ad}_{\lambda(t)}\, y$ fixes e; we refer to ρ as *Slodowy's action*. In order to see how it leads to quasi-homogeneous coordinates, let us denote for $x \in \mathfrak{g}$ by $\mathcal{F}_x$, the function defined by $\mathcal{F}_x(y) := \langle y - e \,|\, x \rangle$, for $y \in \mathfrak{g}$. Then (9) and Ad-invariance of the Killing form imply that if $x \in \mathfrak{g}(i)$ then

$$\begin{aligned} (\rho_t^* \mathcal{F}_x)(y) &= \langle \rho_{t^{-1}} \cdot y - e \,|\, x \rangle = t^{-2} \left\langle \mathrm{Ad}_{\lambda(t^{-1})}(y - e) \,|\, x \right\rangle \\ &= t^{-2} \left\langle y - e \,|\, \mathrm{Ad}_{\lambda(t)}\, x \right\rangle = t^{-2} \left\langle y - e \,|\, t^{-i} x \right\rangle = t^{-i-2} \mathcal{F}_x(y). \end{aligned}$$

It follows that the quasi-degree $\varpi(\mathcal{F}_x)$ of $\mathcal{F}_x$ is $i+2$, for $x \in \mathfrak{g}(i)$. According to (2), one has, for any x, y, $z \in \mathfrak{g}$,

$$\{\mathcal{F}_x, \mathcal{F}_y\}(z) = \langle z \,|\, [x, y] \rangle = \mathcal{F}_{[x,y]}(z) + \langle e \,|\, [x, y] \rangle. \tag{10}$$

If $x \in \mathfrak{g}(i)$ and $y \in \mathfrak{g}(j)$, with $i + j \neq -2$ then $\langle e \,|\, [x, y] \rangle = 0$ and so

$$\begin{aligned} \varpi(\{\mathcal{F}_x, \mathcal{F}_y\}) - \varpi(\mathcal{F}_x) - \varpi(\mathcal{F}_y) &= \varpi(\mathcal{F}_{[x,y]}) - \varpi(\mathcal{F}_x) - \varpi(\mathcal{F}_y) \\ &= i + j + 2 - (i + 2) - (j + 2) = -2. \end{aligned}$$

This result extends to the case $i + j = -2$, since then $\varpi(\mathcal{F}_{[x,y]}) = i + j + 2 = 0$, which is the quasi-degree of the constant function $\langle e \,|\, [x, y] \rangle$. This proves the following proposition.

Proposition 3.2. *Let $\mathfrak{g}$ be a semi-simple Lie algebra, identified with its dual using its Killing form, let $\mathcal{O}$ be a nilpotent adjoint orbit of $\mathfrak{g}$, with canonical triple (h, e, f). Let $x_1, \ldots, x_d$ be any basis of $\mathfrak{g}$, where each x_k belongs to some eigenspace $\mathfrak{g}(i_k)$ of ad_h and let $\mathcal{F}_k$ be the dual coordinates on $\mathfrak{g}$, centered at e, $\mathcal{F}_k(y) := \langle y - e \,|\, x_k \rangle$. Then the Lie-Poisson structure $\{\cdot\,,\cdot\}$ on $\mathfrak{g}$ is quasi-homogeneous of degree -2 with respect to $(\varpi(\mathcal{F}_1), \ldots, \varpi(\mathcal{F}_d)) = (i_1 + 2, \ldots, i_d + 2)$.*

□

We now wish to show that, upon picking a suitable transverse slice N to $\mathcal{O}$ at e, the transverse Poisson structure on N is also quasi-homogeneous (of degree -2). Following [10] we consider the set $\mathcal{N}_h$ of all subspaces $\mathfrak{n}$ of $\mathfrak{g}$ that are complementary to $\mathfrak{g}(e)$ in $\mathfrak{g}$, and which are ad_h-invariant. For $\mathfrak{n}\in\mathcal{N}_h$ we let $N:=e+\mathfrak{n}^\perp$, which is a transverse slice to $\mathbf{G}\cdot e$. The ad_h-invariance of $\mathfrak{n}$ implies on the one hand that ρ leaves N invariant: if $y\in e+\mathfrak{n}^\perp$ then

$$0=\left\langle y-e\,|\;\mathrm{Ad}_{\lambda(t^{-1})}\,\mathfrak{n}\right\rangle=\left\langle \mathrm{Ad}_{\lambda(t)}(y-e)\,|\,\mathfrak{n}\right\rangle=t^{-2}\left\langle \rho_t\cdot y-e\,|\,\mathfrak{n}\right\rangle,$$

so that indeed $\rho_t\cdot y\in e+\mathfrak{n}^\perp$. On the other hand, it implies that $\mathfrak{n}$ admits a basis where each basis vector belongs to an eigenspace of $\mathfrak{h}$. Thus we can specialize the above basis $x_1,\dots,x_d$ so that it be adapted to $\mathfrak{n}$: we can choose a basis $(Z_1,\dots,Z_k)$ for $\mathfrak{g}(e)$ and a basis $(X_1,\dots,X_{2r})$ for $\mathfrak{n}$ in such a way that:

(1) each $Z_i, 1\leq i\leq k$, is a highest weight vector of weight n_i ;
(2) each $X_i, 1\leq i\leq 2r$, is a weight vector of weight ν_i.

The linear coordinates (centered at e) $\mathcal{F}_{Z_1},\dots,\mathcal{F}_{Z_k}$, restricted to N, will be denoted by $q_1,\dots,q_k$. In view of the above, their quasi-degrees are defined as $\varpi(q_i):=n_i+2$. The fact that the transverse Poisson structure is polynomial in terms of these coordinates was first shown in [10, Thm 2.3]. In the following proposition we give a refinement of this statement.

Proposition 3.3. *In the notation of Proposition 3.2, the transverse Poisson structure on $N:=e+\mathfrak{n}^\perp$, where $\mathfrak{n}\in\mathcal{N}$, is a polynomial Poisson structure that is quasi-homogeneous of degree -2, with respect to the quasi-degrees $n_1+2,\dots,n_k+2$, where $n_1,\dots,n_k$ denote the highest weights of $\mathfrak{g}$ as an $\mathfrak{s}$-module.*

Proof. According to (4) we need to show that for any $1\leq i,j\leq k$ the functions A_{ij} and $(BC^{-1}B^\top)_{ij}$ are quasi-homogeneous of degree $\varpi(q_i)+\varpi(q_j)-2=n_i+n_j+2$. For A_{ij} this is clear, since A is part of the Poisson matrix of the Lie-Poisson structure on $\mathfrak{g}$, which we have seen to be quasi-homogeneous of degree -2. Similarly, we have that $\varpi(B_{ip})=n_i+\nu_p+2$. Since

$$\varpi(B_{ip}C^{-1}_{ps}B_{js})=n_i+n_j+\nu_p+\nu_s+4+\varpi(C^{-1}_{ps}),$$

it means that we need to show that

$$\varpi(C^{-1}_{ps})=-\nu_p-\nu_s-2. \tag{11}$$

This follows from the fact that $\sum_{i=1}^{2r}(\nu_i+1)=0$, itself a consequence of (8). Indeed, consider a term of the form $C'_{ij}=C_{i_1j_1}\dots C_{i_{2r-1}j_{2r-1}}$, where

$$\{i_1,i_2,\dots,i_{2r-1}\}=\{1,2,\dots,2r\}\setminus\{s\}\ \text{ and }\ \{j_1,i_2,\dots,j_{2r-1}\}=\{1,2,\dots,2r\}\setminus\{p\}.$$

Then

$$\varpi(C'_{ij})=\sum_{k=1}^{2r-1}(\nu_{i_k}+\nu_{j_k}+2)=-\nu_s-\nu_p-2,$$

A typical term of C^{-1}_{ps} is of the form $\frac{C'_{ij}}{\Delta(C)}$, where $\Delta(C)$ is the determinant of C which is constant, as it is of quasi-degree zero, by the same argument (this observation was already made in [10, Thm 2.3]). This gives us (11). □

Let us consider now the case of any adjoint orbit $\mathbf{G}\cdot x$ and $x=s+e$ the Jordan-Chevalley decomposition of x, a case that we already considered in Proposition 2.1. A well-known result ([15, par. 32.1.7.]) says that there exists a $\mathfrak{sl}_2$-triple (h, e, f) such that $[s, h] = [s, f] = 0$. Consequently, (h, e, f) is a $\mathfrak{sl}_2$-triple of the reductive Lie algebra $\mathfrak{g}(s)$ and we can also suppose that, up to conjugation by elements of $\mathbf{G}(s)$, h is the characteristic of $\mathbf{G}(s) \cdot e$. Let $\mathcal{N}_{s,h}$ the set of all complementary subspaces to $\mathfrak{g}(x)$ in $\mathfrak{g}(s)$ which are ad_h-invariant. Then, by applying Proposition 3.3, we get the following result.

Corollary 3.4. *As in Proposition 2.1, let* $\mathfrak{n}_s = \mathfrak{g}(s)^\perp, \mathfrak{n}_e \in \mathcal{N}_{s,h}$ *and* $\mathfrak{n} = \mathfrak{n}_s \oplus \mathfrak{n}_e$. *Let* $N_x := x + \mathfrak{n}^\perp$, *which is a transverse slice to* $\mathbf{G} \cdot x$. *Then, the transverse Poisson structure on* N_x *is polynomial and is quasi-homogeneous of quasi-degree* -2.

From now, a transverse Poisson structure given by Proposition 3.3 will be called an *adjoint transverse Poisson structure* or ATP-structure.

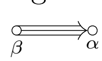
4. Examples

In this section we want to show in two examples how one computes the ATP-structure in practice. In the first example we consider the subregular orbit of $\mathfrak{g}_2$, and we do the computations without chosing a representation of $\mathfrak{g}_2$. In the second example, the subregular orbit of $\mathfrak{so}_8$, we use a concrete representation, rather than referring to tables for the explicit formulas of the Lie brackets in a Chevalley basis. These two examples will also serve later in the paper as an illustration of the results that we will prove on the nature of the ATP-structure. Both examples correspond to subregular orbits, which lead to two of the simplest non-trivial ATP-structures, in the following sense. If $\mathcal{O}$ is an adjoint orbit in $\mathfrak{g}$ then the ATP-structure to $\mathcal{O}$ has rank $\dim \mathfrak{g} - \ell - \dim \mathcal{O}$ at a generic point of any transverse slice to $\mathcal{O}$, since the Lie-Poisson structure on $\mathfrak{g}$ has rank[1] $\dim \mathfrak{g} - \ell$, at a generic point of $\mathfrak{g}$. For the regular nilpotent orbit $\mathcal{O}_{reg}$, the ATP-structure is trivial, because $\dim \mathcal{O}_{reg} = \dim \mathfrak{g} - \ell$. So, the first interesting nilpotent orbit to consider is the subregular orbit, denoted by $\mathcal{O}_{sr}$. We recall two well-known facts (see [5]):

- the subregular orbit $\mathcal{O}_{sr}$ is the unique nilpotent orbit which is open and dense in the complement of $\mathcal{O}_{reg}$ in the nilpotent cone;
- $\dim \mathcal{O}_{sr} = \dim \mathfrak{g} - \ell - 2$.

It follows that the ATP-structure of the subregular orbit is of dimension $\ell + 2$ and its generic rank is 2. In each of the two examples that follow we give the characteristic triplet (h, e, f) that corresponds to the orbit, we derive from it a basis of the ad_h-weight spaces, which leads to basis vectors Z_i of $\mathfrak{g}(e)$ and X_j of an ad_h-invariant complement to $\mathfrak{g}(e)$ in $\mathfrak{g}$. The Lie brackets of these elements then lead to the matrices A, B and C in (3), which by Dirac's formula (4) yields the matrix Λ_N of the transverse Poisson structure.

4.1. **The subregular orbit of type G_2.** We first consider the case of the subregular orbit of the Lie algebra $\mathfrak{g} := \mathfrak{g}_2$. Denoting the basis of simple roots by $\Pi = \{\alpha, \beta\}$, where β is the longer root, its Dynkin diagram is given by



and it has the following positive roots : $\Delta_+ = \{\alpha, \beta, \alpha + \beta, 2\alpha + \beta, 3\alpha + \beta, 3\alpha + 2\beta, 3\alpha + \beta, 3\alpha + 2\beta\}$. The vectors that constitute the Chevalley basis[2] of $\mathfrak{g}$ are denoted by H_α, H_β for the Cartan subalgebra, X_γ for the 6 positive roots $\gamma \in \Delta_+$ and Y_γ for

[1] Recall that ℓ denotes the rank of $\mathfrak{g}$.

[2] The choice of Chevalley basis that we use is explicitly described in [14, Chapter VII.4].

the six negative roots $-\gamma \in \Delta_+$. According to [5, Chap. 8.4], the characteristic h of the subregular orbit $\mathcal{O}_{sr}$ is given by the sequence of weights $(0,2)$, which means that $\langle \alpha, h\rangle = 0$ and $\langle \beta, h\rangle = 2$, which yields $h = 2H_\alpha + 4H_\beta$. The decomposition of $\mathfrak{g}$ into ad_h-weight spaces $\mathfrak{g}(i)$ consists of the following five subspaces

$$\begin{aligned}
\mathfrak{g}(4) &= \langle X_{3\alpha+2\beta}\rangle, \\
\mathfrak{g}(2) &= \langle X_\beta,\, X_{\alpha+\beta},\, X_{2\alpha+\beta},\, X_{3\alpha+\beta}\rangle, \\
\mathfrak{g}(0) &= \langle H_\alpha,\, H_\beta,\, X_\alpha,\, Y_\alpha\rangle, \\
\mathfrak{g}(-2) &= \langle Y_\beta,\, Y_{\alpha+\beta},\, Y_{2\alpha+\beta},\, Y_{3\alpha+\beta}\rangle, \\
\mathfrak{g}(-4) &= \langle Y_{3\alpha+2\beta}\rangle.
\end{aligned} \tag{12}$$

Taking for e and f an arbitrary linear combination of the above basis elements of $\mathfrak{g}(2)$, resp. of $\mathfrak{g}(-2)$ and expressing that $[e,f] = h$ one easily finds that the $\mathfrak{sl}_2$-triple corresponding to $\mathcal{O}_{sr}$ is given by

$$e = X_\beta + X_{3\alpha+\beta},\ h = 2H_\alpha + 4H_\beta,\ f = 2Y_\beta + 2Y_{3\alpha+\beta}.$$

Picking the vectors in the positive subspaces $\mathfrak{g}(i)$ that commute with e leads to the following basis vectors of $\mathfrak{g}(e)$:

$$\begin{aligned}
Z_1 &= X_\beta + X_{3\alpha+\beta}. \\
Z_2 &= X_{2\alpha+\beta}, \\
Z_3 &= X_{\alpha+\beta}, \\
Z_4 &= X_{3\alpha+2\beta},
\end{aligned} \tag{13}$$

We obtain an ad_h-invariant complementary subspace $\mathfrak{n}$ of $\mathfrak{g}(e)$ by completing these vectors with additional vectors that are taken from the bases (12) of the subspaces $\mathfrak{g}(i)$. Our choice of basis vectors for $\mathfrak{n}$, ordered by weight, is as follows:

$$\begin{aligned}
X_1 &= X_\beta, & X_6 &= Y_\beta, \\
X_2 &= X_\alpha, & X_7 &= Y_{\alpha+\beta}, \\
X_3 &= H_\alpha, & X_8 &= Y_{2\alpha+\beta}, \\
X_4 &= H_\beta, & X_9 &= Y_{3\alpha+\beta}, \\
X_5 &= Y_\alpha, & X_{10} &= Y_{3\alpha+2\beta}.
\end{aligned}$$

The Lie brackets of these basis vectors for $\mathfrak{g}$, which are listed in [14, Chap. VII.4], yield the Poisson matrix $\begin{pmatrix} A & B \\ -B^\top & C \end{pmatrix}$ of the Lie-Poisson structure on $\mathfrak{g}$, in terms of the coordinates $\mathcal{F}_{Z_1}, \dots, \mathcal{F}_{Z_4}, \mathcal{F}_{X_1}, \dots, \mathcal{F}_{X_{10}}$ on $\mathfrak{g}$, as $A_{ij} = \left\{\mathcal{F}_{Z_i}, \mathcal{F}_{Z_j}\right\} = \mathcal{F}_{[Z_i,Z_j]} + \langle e \,|\, [Z_i, Z_j]\rangle$ (see (10)), and similarly for the other elements of the Poisson matrix. We give the restriction of the matrices A, B and C to the transverse slice $N := e + \mathfrak{n}^\perp$ only, which amounts to keeping in the Lie brackets only the vectors $Z_1, \dots, Z_4$, as $\mathcal{F}_X(n) = \langle e - n \,|\, X\rangle = 0$ for $X \in \mathfrak{n}$ and $n \in N = e + \mathfrak{n}^\perp$. In terms of the coordinates $q_1, \dots, q_4$ on N, where q_i is the restriction of $\mathcal{F}_{Z_i}$ to N, we get

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -3q_4 & 0 \\ 0 & 3q_4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 & 0 & -q_4 & 0 & q_1 & -q_2 & q_3 & 0 & 0 \\ 0 & 3q_1 & -q_2 & 0 & 2q_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2q_2 & q_3 & -q_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ q_4 & q_3 & -3q_1 & q_1 & q_2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$C = \frac{1}{3}\begin{pmatrix} 0 & 3q_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -3q_3 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & -3 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Substituted in (4) this yields the following Poisson matrix for the ATP-structure:

$$\Lambda_N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -3q_4 & 2q_1q_2 - 2q_3^2 \\ 0 & 3q_4 & 0 & 2q_2^2 - 2q_1q_3 \\ 0 & -2q_1q_2 + 2q_3^2 & -2q_2^2 + 2q_1q_3 & 0 \end{pmatrix}. \tag{14}$$

It follows from (12) and (13) that the quasi-degree of q_1, q_2 and q_3 is 4, while the quasi-degree of q_4 is 6. One easily reads off from (14) that, with respect to these quasi-degrees, the ATP-structure is quasi-homogeneous of quasi-degree -2.

4.2. **The subregular orbit of type D_4.** We take now $\mathfrak{g} = \mathfrak{so}_8$ and we realize $\mathfrak{g}$ as the following set of matrices :

$$\left\{ \begin{pmatrix} Z_1 & Z_2 \\ Z_3 & -Z_1^\top \end{pmatrix} \mid Z_i \in \mathrm{Mat}_4(\mathbf{C}),\ \text{with } Z_2, Z_3 \text{ skew-symmetric} \right\}.$$

Let $\mathfrak{h}$ denote the Cartan subalgebra of $\mathfrak{g}$ that consists of all diagonal matrices in $\mathfrak{g}$. It is clear that $\mathfrak{h}$ is spanned by the four matrices $H_i := E_{i,i} - E_{4+i,4+i}$, $1 \le i \le 4$. Define for $i = 1, \dots 4$ the linear map $e_i \in \mathfrak{h}^*$ by $e_i(\sum a_k H_k) = a_i$. Then the root system of $\mathfrak{g}$ is:

$$\Delta := \{\pm e_i \pm e_j \mid 1 \le i, j \le 4, i \ne j\},$$

and a basis of simple roots is $\Pi := \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, where

$$\alpha_1 = e_1 - e_2,\ \alpha_2 = e_2 - e_3,\ \alpha_3 = e_3 - e_4,\ \alpha_4 = e_3 + e_4.$$

It leads to the following Chevalley basis of $\mathfrak{g}$:

$$\begin{aligned} X_{e_i - e_j} &= E_{i,j} - E_{4+j,4+i}, \\ X_{e_i + e_j} &= E_{i,4+j} - E_{j,4+i}, \qquad i < j \\ X_{-e_i - e_j} &= -E_{4+i,j} + E_{4+j,i}, \qquad i < j \\ H_{e_i - e_j} &= H_i - H_j, \\ H_{e_i + e_j} &= H_i + H_j. \end{aligned}$$

According to ([5, Chap. 5.4]), the characteristic h of the subregular orbit is given by the sequence of weights $(2, 0, 2, 2)$. It follows that

$$h = 4H_{\alpha_1} + 6H_{\alpha_2} + 4H_{\alpha_3} + 4H_{\alpha_4}.$$

The positive ad_h-weight spaces are

$$
\begin{aligned}
\mathfrak{g}(0) &= \mathfrak{h} \oplus \langle X_{\alpha_2}, X_{-\alpha_2} \rangle, \\
\mathfrak{g}(2) &= \langle X_{\alpha_1}, X_{\alpha_3}, X_{\alpha_4}, X_{\alpha_1+\alpha_2}, X_{\alpha_2+\alpha_3}, X_{\alpha_2+\alpha_4} \rangle, \\
\mathfrak{g}(4) &= \langle X_{\alpha_1+\alpha_2+\alpha_3}, X_{\alpha_2+\alpha_3+\alpha_4}, X_{\alpha_1+\alpha_2+\alpha_4} \rangle, \\
\mathfrak{g}(6) &= \langle X_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}, X_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4} \rangle.
\end{aligned} \tag{15}
$$

As in the first example it follows that the following is the canonical $\mathfrak{sl}_2$-triple, associated to $\mathcal{O}_{sr}$:

$$
\begin{aligned}
e &= X_{\alpha_1} + X_{\alpha_1+\alpha_2} - X_{\alpha_2+\alpha_4} + 2X_{\alpha_3} - X_{\alpha_4}, \\
h &= 4H_{\alpha_1} + 6H_{\alpha_2} + 4H_{\alpha_3} + 4H_{\alpha_4}, \\
f &= X_{-\alpha_1} + 3X_{-\alpha_1-\alpha_2} - 3X_{-\alpha_2-\alpha_4} + 2X_{-\alpha_3} - X_{-\alpha_4}.
\end{aligned}
$$

We can now define the basis vectors Z_i of $\mathfrak{g}(e)$ and X_j of an ad_h-invariant complementary subspace $\mathfrak{n}$ to $\mathfrak{g}(e)$, in terms of the Chevalley basis, as follows:

$$
\begin{aligned}
Z_1 &= X_{\alpha_1+\alpha_2} - X_{\alpha_2+\alpha_4} + 2X_{\alpha_3}, \\
Z_2 &= X_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}, \\
Z_3 &= X_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}, \\
Z_4 &= X_{\alpha_1} - X_{\alpha_4}, \\
Z_5 &= X_{\alpha_2+\alpha_3} + X_{\alpha_2+\alpha_4} - X_{\alpha_3} - X_{\alpha_4}, \\
Z_6 &= X_{\alpha_1+\alpha_2+\alpha_3} + X_{\alpha_1+\alpha_2+\alpha_4} - X_{\alpha_2+\alpha_3+\alpha_4},
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
X_1 &= X_{\alpha_1+\alpha_2+\alpha_3}, & X_{12} &= X_{-\alpha_1}, \\
X_2 &= X_{\alpha_2+\alpha_3+\alpha_4}, & X_{13} &= X_{-\alpha_3}, \\
X_3 &= X_{\alpha_4} & X_{14}, &= -X_{-\alpha_4}, \\
X_4 &= X_{\alpha_3}, & X_{15} &= X_{-\alpha_1-\alpha_2}, \\
X_5 &= X_{\alpha_2+\alpha_4}, & X_{16} &= X_{-\alpha_2-\alpha_3}, \\
X_6 &= H_{\alpha_1}, & X_{17} &= -X_{-\alpha_2-\alpha_4}, \\
X_7 &= H_{\alpha_2}, & X_{18} &= -X_{-\alpha_1-\alpha_2-\alpha_3}, \\
X_8 &= H_{\alpha_3}, & X_{19} &= -X_{-\alpha_1-\alpha_2-\alpha_4}, \\
X_9 &= H_{\alpha_4}, & X_{20} &= -X_{-\alpha_2-\alpha_3-\alpha_4}, \\
X_{10} &= X_{\alpha_2}, & X_{21} &= -X_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4}, \\
X_{11} &= X_{-\alpha_2}, & X_{22} &= -X_{-\alpha_1-2\alpha_2-\alpha_3-\alpha_4}.
\end{aligned}
$$

If we denote by $\bar{Z}_1, \ldots, \bar{Z}_6$ the dual basis (w.r.t. $\langle X \,|\, Y \rangle = \frac{1}{2}\,\mathrm{Trace}(XY)$) of the basis $Z_1, \ldots, Z_6$ of $\mathfrak{g}(e)$ then a typical element of the transverse slice $N = e + \mathfrak{n}^{\perp}$ is given by

$e+\sum_{i=1}^{6} q_i\bar{Z}_i$, i.e.,

$$Q=\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0\\ q_4 & 0 & 0 & 0 & 0 & 0 & 0 & -1\\ q_1 & 0 & 0 & 2 & 0 & 0 & 0 & -1\\ 0 & q_5 & 0 & 0 & 0 & 1 & 1 & 0\\ 0 & -q_3 & -q_2 & 0 & 0 & -q_4 & -q_1 & 0\\ q_3 & 0 & q_6 & 0 & -1 & 0 & 0 & -q_5\\ q_2 & -q_6 & 0 & 0 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \end{pmatrix} \tag{17}$$

and we can compute the matrix A, restricted to N, by $A_{ij}=\langle Q\,|\,[Z_i,Z_j]\rangle$, and similarly for the matrices B and C. A direct substitution in (4) leads to the following Poisson matrix for the the ATP-structure:

$$\Lambda_N=\frac{1}{2}\begin{pmatrix} 0 & q_4q_6 & -q_4q_6 & 0 & -2q_6 & 2q_{16}\\ -q_4q_6 & 0 & 0 & q_4q_6 & -q_5q_6 & -2q_{36}\\ q_4q_6 & 0 & 0 & -q_4q_6 & q_5q_6 & 2q_{36}\\ 0 & -q_4q_6 & q_4q_6 & 0 & 2q_6 & -2q_{16}\\ 2q_6 & q_5q_6 & -q_5q_6 & -2q_6 & 0 & 2q_{56}\\ -2q_{16} & 2q_{36} & -2q_{36} & 2q_{16} & -2q_{56} & 0 \end{pmatrix}, \tag{18}$$

where

$$\begin{aligned} q_{16} &= 2q_2-q_1q_4-q_4q_5+q_4^2,\\ q_{36} &= q_3q_4-q_2q_4-q_2q_5,\\ q_{56} &= 2q_3-2q_2-q_5^2+q_4q_5-q_1q_5. \end{aligned} \tag{19}$$

It follows from (15) and (16) that the quasi-degrees of the variables q_i are: $\varpi(q_1)=\varpi(q_4)=\varpi(q_5)=4$, $\varpi(q_2)=\varpi(q_3)=8$ and $\varpi(q_6)=6$. The fact that the ATP-structure is quasi-homogeneous of quasi-degree -2 is again easily read off from (18).

5. The subregular case

In this section we will give an explicit description of the ATP-structure in the case of the subregular orbit $\mathcal{O}_{sr}\subset\mathfrak{g}$, where $\mathfrak{g}$ is a semi-simple Lie algebra. Since in the case of the subregular orbit the generic rank of the ATP-structure on the transverse slice N is two and since we know $\dim(N)-2$ independent Casimirs, namely the basic Ad-invariant functions on $\mathfrak{g}$, restricted to N, we will easily derive that the ATP-structure is the determinantal structure (also called Nambu structure), determined by these Casimirs, up to multiplication by a *function.* What is much less trivial to show is that this function is actually just a *constant.* For this we will use Brieskorn's theory of simple singularities, which is recalled in Paragraph 5.2 below. First we recall the basic facts on Ad-invariant functions on $\mathfrak{g}$ and link them to the ATP-structure.

5.1. **Invariant functions and Casimirs.** Let $\mathcal{O}_{sr}=\mathbf{G}\cdot e$, be a subregular orbit in the semi-simple Lie algebra $\mathfrak{g}$, let (h,e,f) be the corresponding canonical $\mathfrak{sl}_2$-triple and consider the transverse slice $N:=e+\mathfrak{n}^\perp$ to $\mathbf{G}\cdot e$, where $\mathfrak{n}$ is an ad_h-invariant complement to $\mathfrak{g}(e)$. We know from Section 3 that the ATP-structure on N, equipped with the linear coordinates $q_1,\dots,q_k$, is a quasi-homogeneous polynomial Poisson structure of generic rank 2. Let $S(\mathfrak{g}^*)^{\mathbf{G}}$ be the algebra of Ad-invariant polynomial functions on $\mathfrak{g}$. By a classical theorem due to Chevalley, $S(\mathfrak{g}^*)^{\mathbf{G}}$ is a polynomial algebra, generated by ℓ homogeneous polynomials $(G_1,\dots,G_\ell)$, whose degree $d_i:=\deg(G_i)=m_i+1$, where

$m_1, \ldots, m_\ell$ are the exponents of $\mathfrak{g}$. These functions are Casimirs of the Lie-Poisson structure on $\mathfrak{g}$, since Ad-invariance of G_i implies that $[x, dG_i(x)] = 0$ and hence the Lie-Poisson bracket (2) is given by

$$\{F, G_i\}(x) = \langle x \mid [dF(x), dG_i(x)] \rangle = -\langle [x, dG_i(x)] \mid dF(x) \rangle = 0,$$

for any function F on $\mathfrak{g}$. If we denote by χ_i the restriction of G_i to the transverse slice N then it follows that these functions are Casimirs of the ATP-structure. The polynomials χ_i are not homogeneous, but they are quasi-homogeneous, as shown in the following proposition.

Lemma 5.1. *Each χ_i is a quasi-homogeneous polynomial of quasi-degree $2d_i$, relative to the quasi-degrees $(2+n_1, \ldots, 2+n_k)$.*

Proof. Since χ_i is of degree d_i and χ_i is Ad-invariant, we get

$$\rho_t^*(\chi_i) = \chi_i \circ \rho_{t^{-1}} = \chi_i \circ (t^{-2} \operatorname{Ad}_{\lambda^{-1}(t)}) = t^{-2d_i} \chi_i \circ \operatorname{Ad}_{\lambda^{-1}(t)} = t^{-2d_i} \chi_i,$$

so that χ_i has quasi-degree $2d_i$. $\square$

5.2. **Simple singularities.** Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. The Weyl group $\mathcal{W}$ acts on $\mathfrak{h}$ and the algebra $S(\mathfrak{g}^*)^{\mathbf{G}}$ of Ad-invariant polynomial functions on $\mathfrak{g}$ is isomorphic to $S(\mathfrak{h}^*)^{\mathcal{W}}$, the algebra of $\mathcal{W}$-invariant polynomial functions on $\mathfrak{h}^*$. The inclusion homomorphism $S(\mathfrak{g}^*)^{\mathbf{G}} \hookrightarrow S(\mathfrak{g}^*)$, is dual to a morphism $\mathfrak{g} \to \mathfrak{h}/\mathcal{W}$, called the *adjoint quotient*. Concretely, the adjoint quotient is given by

$$\begin{array}{rcccl} G & : & \mathfrak{g} & \to & \mathbf{C}^\ell \\ & & x & \mapsto & (G_1(x),\, G_2(x), \ldots, G_\ell(x)). \end{array} \tag{20}$$

The zero-fiber $G^{-1}(0)$ of G is exactly the nilpotent variety $\mathcal{N}$ of $\mathfrak{g}$. As we are interested in $N \cap \mathcal{N} = N \cap G^{-1}(0) = \chi^{-1}(0)$, which is an affine surface with an isolated, simple singularity, let us recall the notion of simple singularity (see [12] for details). Up to conjugacy, there are five types of finite subgroups of $\mathbf{SL}_2 = \mathbf{SL}_2(\mathbf{C})$, denoted by $\mathcal{C}_p, \mathcal{D}_p, \mathcal{T}, \mathcal{O}$ and $\mathcal{I}$. Given such a subgroup $\mathbf{F}$, one looks at the corresponding ring of invariant polynomials $\mathbf{C}[U,V]^{\mathbf{F}}$. In each of the five cases, $\mathbf{C}[u,v]^{\mathbf{F}}$ is generated by three fundamental polynomials X, Y, Z, subject to only one relation $R(X,Y,Z) = 0$, hence the quotient space $\mathbf{C}^2/\mathbf{F}$ can be identified, as an affine surface, with the singular surface in $\mathbf{C}^3$, defined by $R = 0$. The origin is its only singular point; it is called a *(homogeneous) simple singularity.* The exceptional divisor of the minimal resolution of $\mathbf{C}^2/\mathbf{F}$ is a finite set of projective lines. If two of these lines meet, then they meet in a single point, and transversally. Moreover, the intersection pattern of these lines forms a graph that coincides with one of the simply laced Dynkin diagrams of type A_ℓ, D_ℓ, E_6, E_7 or E_8. This type is called the type of the singularity. Moreover, every such Dynkin diagram (i.e., of type ADE) appears in this way; see Table 1.

For the other simple Lie algebras (of type B_ℓ, C_ℓ, F_4 or G_2), there exists a similar correspondence. By definition, an *(inhomogeneous) simple singularity* of type Δ is a couple (V, Γ) consisting of a homogeneous simple singularity $V = \mathbf{C}^2/\mathbf{F}$ and a group $\Gamma = \mathbf{F}'/\mathbf{F}$ of automorphisms of V according to Table 2.

The connection between the diagram of (V, Γ) and that of V can be described as follows: the action of Γ on V lifts to an action on a minimal resolution of V which permutes the components of the exceptional set. Then, we obtain the diagram of (V, Γ) as a Γ-quotient of that of V. It leads to Table 3, which is the non-simply laced analog of Table 1.

TABLE 1. The basic correspondence between finite subgroups $\mathbf{F}$ of $\mathbf{SL}_2$, homogeneous simple singularities, defined by an equation $R(X,Y,Z)=0$ and simply laced simple Lie algebras of type Δ.

Group $\mathbf{F}$	Singularity $R(X,Y,Z)=0$	Type Δ
$\mathcal{C}_{\ell+1}$	$X^{\ell+1}+YZ=0$	A_ℓ
$\mathcal{D}_{\ell-2}$	$X^{\ell-1}+XY^2+Z^2=0$	D_ℓ
$\mathcal{T}$	$X^4+Y^3+Z^2=0$	E_6
$\mathcal{O}$	$X^3Y+Y^3+Z^2=0$	E_7
$\mathcal{I}$	$X^5+Y^3+Z^2=0$	E_8

TABLE 2. We give the list of all possible inhomogeneous singularities of type $\Delta=(V,\Gamma)$, where V is one of the homogeneous simple singularities and $\Gamma=\mathbf{F}'/\mathbf{F}$ is a group of automorphisms of V. The labels B_ℓ, C_ℓ, F_4 and G_2 for these types will become clear in Proposition 5.2.

Type Δ	V	$\mathbf{F}$	$\mathbf{F}'$	$\Gamma=\mathbf{F}'/\mathbf{F}$
B_ℓ	$A_{2\ell-1}$	$\mathcal{C}_{2\ell}$	$\mathcal{D}_\ell$	$\mathbf{Z}/2\mathbf{Z}$
C_ℓ	$D_{\ell+1}$	$\mathcal{D}_{\ell-1}$	$\mathcal{D}_{2\ell-2}$	$\mathbf{Z}/2\mathbf{Z}$
F_4	E_6	$\mathcal{T}$	$\mathcal{O}$	$\mathbf{Z}/2\mathbf{Z}$
G_2	D_4	$\mathcal{D}_2$	$\mathcal{O}$	$\mathbf{Z}/3\mathbf{Z}$

We can now state the following extension of a theorem of Brieskorn, which is due to Slodowy ([12, Thms 1 and 2]).

Proposition 5.2. *Let $\mathfrak{g}$ be a simple complex Lie algebra, with Dynkin diagram of type Δ. Let $\mathcal{O}_{sr}=\mathbf{G}\cdot e$ be the subregular orbit and $N=e+\mathfrak{n}^\perp$ a transverse slice to $\mathbf{G}\cdot e$. The surface $N\cap\mathcal{N}=\chi^{-1}(0)$ has a (homogeneous or inhomogeneous) simple singularity of type Δ.*

To finish this paragraph, we illustrate the above results in the case of the two examples that were given in Section 4. In both cases we give the invariants, restricted to the slice N, and their zero locus, the surface $\chi^{-1}(0)$.

1) For the subregular orbit of $\mathfrak{g}_2$, the invariant functions, restricted to the slice N, are given by

$$\begin{aligned} \chi_1 &= q_1, \\ \chi_2 &= 12q_1q_2q_3-4q_2^3-4q_3^3+9q_4^2, \end{aligned} \tag{21}$$

TABLE 3. For each of the inhomogeneous simple singularities, of type Δ (see Table 2), the correspondending homogeneous simple singularity $V = \mathbf{C}^2/\mathbf{F}$ is given by its equation $R(X,Y,Z) = 0$, together with the action of $\Gamma = \mathbf{F}'/\mathbf{F}$ on V. In the last line, α is a non-trivial cubic root of unity.

Type Δ	Singularity $R(X,Y,Z)=0$	Γ-action
B_ℓ	$X^{2\ell}+YZ=0$	$(X,Y,Z)\longrightarrow(-X,Z,Y)$
C_ℓ	$X^\ell+XY^2+Z^2=0$	$(X,Y,Z)\longrightarrow(X,-Y,-Z)$
F_4	$X^4+Y^3+Z^2=0$	$(X,Y,Z)\longrightarrow(-X,Y,-Z)$
G_2	$X^3+Y^3+Z^2=0$	$(X,Y,Z)\longrightarrow(\alpha X,\alpha^2 Y,Z)$

which leads to an affine surface $\chi^{-1}(0)$ in $\mathbf{C}^4$, isomorphic to the surface in $\mathbf{C}^3$ defined by

$$4q_2^3 + 4q_3^3 - 9q_4^2 = 0.$$

Up to a rescaling of the coordinates, this is the polynomial R that was given in Table 3.

2) For the subregular orbit of $\mathfrak{so}_8$, the invariant functions, restricted to the slice N, are found as the (non-constant) coefficients of the characteristic polynomial of the matrix Q (see (17)):

$$\begin{aligned}
\chi_1 &= -2q_1 - 2q_4,\\
\chi_2 &= -12q_2 - 4q_3 - 4q_4q_5 + (q_1+q_4)^2,\\
\chi_3 &= -q_2 + q_3 - q_4q_5,\\
\chi_4 &= -4q_1q_2 - 16q_2q_5 - 12q_3q_4 + 12q_2q_4 + 4q_1q_3 + 4q_4^2q_5 + 4q_1q_4q_5 - 4q_6^2.
\end{aligned} \tag{22}$$

By linearly eliminating the variables q_1, q_2 and q_3 from the equations $\chi_i = 0$, for $i = 1,2,3$, we find that $\chi^{-1}(0)$ is isomorphic to the affine surface in $\mathbf{C}^3$, defined by

$$4q_4^2q_5 - 2q_4q_5^2 + q_6^2 = 0.$$

Its defining polynomial corresponds to the polynomial R in Table 1, upon putting $X = i\gamma q_4$, $Y = \gamma(q_5 - q_4)$ and $Z = q_6$, where γ is any cubic root of $2i$.

5.3. **The determinantal Poisson structure.** We prove in this paragraph the announced result that the ATP-structure in the subregular case is a determinantal Poisson structure, determined by the Casimirs. Let us first point out how such a determinantal Poisson structure is defined. Let $C_1,\dots,C_{d-2}$ be $d-2$ (algebraically) independent polynomials in $d>2$ variables $x_1,\dots,x_d$. For such a polynomial F, let us denote by ∇F its differential $\mathsf{d}F$, expressed in the natural basis $\mathsf{d}x_i$, i.e., ∇F is a column vector with elements $(\nabla F)_i = \frac{\partial F}{\partial x_i}$. Then a polynomial Poisson structure is defined on $\mathbf{C}^d$ by

$$\{F,G\}_{det} := \det(\nabla F,\ \nabla G,\ \nabla C_1,\ \dots,\ \nabla C_{d-2}), \tag{23}$$

where F and G are arbitrary polynomials. It is clear that each of the C_i is a Casimir of $\{\cdot\,,\cdot\}_{det}$, so that in particular the generic rank of $\{\cdot\,,\cdot\}_{det}$ is two. Notice also that if the Casimirs C_i are quasi-homogeneous, with respect to the weights $\varpi_i := \varpi(x_i)$ then for any quasi-homogeneous elements F and G we have that

$$\varpi(\{F,G\}_{det}) = \varpi(F) + \varpi(G) + \sum_{i=1}^{d-2} \varpi(C_i) - \sum_{i=1}^{d} \varpi_i,$$

an easy consequence of the definition of a determinant and of the following obvious fact: if F is any quasi-homogeneous polynomial, then $\frac{\partial F}{\partial x_i}$ is quasi-homogeneous and $\varpi(\frac{\partial F}{\partial x_i}) = \varpi(F) - \varpi_i$. It follows that $\{\cdot\,,\cdot\}_{det}$ is quasi-homogeneous of quasi-degree κ, where

$$\kappa = \sum_{i=1}^{d-2} \varpi(C_i) - \sum_{i=1}^{d} \varpi_i. \tag{24}$$

Applied to our case it means that we have two polynomial Poisson structures on the transverse slice N which have $G_1, \ldots, G_\ell$ as Casimirs on $N \cong \mathbf{C}^{\ell+2}$, namely the ATP-structure and the determinantal structure, constructed by using these Casimirs.

In our two examples (see Section 4), these structures are easily compared by explicit computation. For the subregular orbit of $\mathfrak{g}_2$, we have according to (23) that

$$(\Lambda_{det})_{ij} = \det\left(\nabla q_i \ \nabla q_j \ \nabla \chi_1 \ \nabla \chi_2\right),$$

where χ_1 and χ_2 are the Casimirs (21). This leads to

$$\Lambda_{det} = -6 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -3q_4 & 2q_1q_2 - 2q_3^2 \\ 0 & 3q_4 & 0 & 2q_2^2 - 2q_1q_3 \\ 0 & -2q_1q_2 + 2q_3^2 & -2q_2^2 + 2q_1q_3 & 0 \end{pmatrix}.$$

In view of (14), it follows that $\Lambda_{det} = -6\Lambda_N$, so that both Poisson structures coincide. For $\mathfrak{so}_8$ one finds similarly, using the Casimirs $\chi_1, \ldots, \chi_4$ in (22)

$$\Lambda_{det} = -128 \begin{pmatrix} 0 & q_4q_6 & -q_4q_6 & 0 & -2q_6 & 2q_{16} \\ -q_4q_6 & 0 & 0 & q_4q_6 & -q_5q_6 & -2q_{36} \\ q_4q_6 & 0 & 0 & -q_4q_6 & q_5q_6 & 2q_{36} \\ 0 & -q_4q_6 & q_4q_6 & 0 & 2q_6 & -2q_{16} \\ 2q_6 & q_5q_6 & -q_5q_6 & -2q_6 & 0 & 2q_{56} \\ -2q_{16} & 2q_{36} & -2q_{36} & 2q_{16} & -2q_{56} & 0 \end{pmatrix},$$

where q_{16}, q_{36} and q_{56} are given by (19). In view of (18), both Poisson structures again coincide, $\Lambda_{det} = -256\Lambda_N$.

In order to show that, in the subregular case, the ATP-structure and the determinantal structure always coincide, i.e., differ by a constant factor only, we first show, that both structures coincide, up to factor, which is a rational function.

Proposition 5.3. *Let $\{\cdot\,,\cdot\}$ and $\{\cdot\,,\cdot\}'$ be two non-trivial polynomial Poisson structures on $\mathbf{C}^d$ which have $d-2$ common independent polynomial Casimirs $C_1, \ldots, C_{d-2}$. There exists a rational function $R \in \mathbf{C}(x_1, \ldots, x_d)$ such that $\{\cdot\,,\cdot\} = R\,\{\cdot\,,\cdot\}'$.*

Proof. Let M and M' denote the Poisson matrices that correspond to $\{\cdot\,,\cdot\}$ and $\{\cdot\,,\cdot\}'$ with respect to the coordinates $x_1, \ldots, x_d$. If we denote $\mathcal{R} := \mathbf{C}(x_1, \ldots, x_d)$ then M and M' both act naturally as a skew-symmetric endomorphism on the $\mathcal{R}$-vector space $\mathcal{R}^d$. The subspace H of $\mathcal{R}^d$ that is spanned by $\nabla C_1, \ldots, \nabla C_{d-2}$ is the kernel of both maps, hence we have two induced skew-symmetric endomorphisms φ and φ' of the quotient space $\mathcal{R}^d/H$. Since the latter is two-dimensional, φ' and φ are proportional, $\varphi' = R\varphi$, where $R \in \mathcal{R}$. Since M and M' have the same kernel, $M' = RM$. $\square$

Applied to our two Poisson structures $\{\cdot\,,\cdot\}_N$ and $\{\cdot\,,\cdot\}_{det}$ the proposition yields that $\{\cdot\,,\cdot\}_N = R\{\cdot\,,\cdot\}_{det}$, where $R = P/Q \in \mathcal{R}$. We show in the following result that R is actually a (non-zero) constant, thereby characterizing completely the ATP-structure in the subregular case.

Theorem 5.4. *Let $\mathcal{O}_{sr}$ be the subregular nilpotent adjoint orbit of a complex semi-simple Lie algebra $\mathfrak{g}$ and let (h,e,f) be the canonical triple, associated to $\mathcal{O}_{sr}$. Let $N = e+\mathfrak{n}^\perp$ be a transverse slice to $\mathcal{O}_{sr}$, where $\mathfrak{n}$ is an ad_h-invariant complementary subspace to $\mathfrak{g}(e)$. Let $\{\cdot\,,\cdot\}_N$ and $\{\cdot\,,\cdot\}_{det}$ denote respectively the ATP-structure and the determinantal structure on N. Then $\{\cdot\,,\cdot\}_N = c\{\cdot\,,\cdot\}_{det}$ for some $c \in \mathbf{C}^*$.*

Proof. By the above, $\{\cdot\,,\cdot\}_N = R\{\cdot\,,\cdot\}_{det}$, where $R \in \mathcal{R}$. If R has a non-trivial denominator Q, then all elements of the Poisson matrix of $\{\cdot\,,\cdot\}_{det}$ must be divisible by Q, since both Poisson structures are polynomial. Then along the hypersurface $Q = 0$ the rank of $(\nabla\chi_1, \dots, \nabla\chi_\ell)$ is smaller than ℓ, hence $\chi^{-1}(0)$ is singular along the curve $\chi^{-1}(0) \cap (Q = 0)$. However, by Proposition 5.2, we know that $\chi^{-1}(0)$ has an isolated singularity, leading to a contradiction. This shows that Q is a constant, and hence that R is a polynomial.

In order to show that the polynomial R is constant it suffices to show that the quasi-degrees of $\{\cdot\,,\cdot\}_N$ and $\{\cdot\,,\cdot\}_{det}$ are the same, which amounts (in view of Proposition 3.2) to showing that the quasi-degree of $\{\cdot\,,\cdot\}_{det}$ is -2. This follows from the following formula, due to Kostant (see [9, Thm 7]), which expresses the dimension of the regular orbit in terms of the exponents m_i of $\mathfrak{g}$:

$$2\sum_{i=1}^{\ell} m_i = \dim \mathcal{O}_{reg} = \dim \mathfrak{g} - \ell. \tag{25}$$

Indeed, if we use this formula, Lemma 5.1 and (8) in the formula (24) for the quasi-degree of $\{\cdot\,,\cdot\}_{det}$, then we find

$$\begin{aligned}
\kappa &= \sum_{i=1}^{\ell} \varpi(\chi_i) - \sum_{i=1}^{\ell+2} \varpi(q_i) = 2\sum_{i=1}^{\ell} d_i - \sum_{i=1}^{\ell+2}(n_i+2)\\
&= 2\sum_{i=1}^{\ell} m_i - \sum_{i=1}^{\ell+2} n_i - 4\\
&= \dim\mathfrak{g} - \ell - (\dim\mathfrak{g} - \ell - 2) - 4 = -2.
\end{aligned}$$

$\square$

5.4. **Reduction to a 3×3 Poisson matrix.** Let $\mathcal{O}_{sr}$ be the subregular nilpotent adjoint orbit of a complex semi-simple Lie algebra $\mathfrak{g}$ of rank ℓ. Let (h,e,f) be its associated canonical $\mathfrak{sl}_2$-triple, and let $N := e+\mathfrak{n}^\perp$ be a transverse slice to $\mathcal{O}_{sr}$, where $\mathfrak{n}$ is an ad_h-invariant complementary subspace to $\mathfrak{g}(e)$. Let $\{\cdot\,,\cdot\}_N$ be the ATP-structure defined on N. Recall that N is equipped with linear coordinates $q_1, \dots, q_{\ell+2}$ defined in Section 2, and that $\{\cdot\,,\cdot\}_N$ has independent Casimirs $\chi_1, \dots, \chi_\ell$, which are the restrictions to N of the basic homogeneous invariant polynomial functions on $\mathfrak{g}$.

Our goal now is to show that, in well-chosen coordinates, the ATP-structure $\{\cdot\,,\cdot\}_N$ on N is essentially given by a 3×3 skew-symmetric matrix, closely related to the polynomial that defines the singularity. More precisely, we have the following theorem.

Theorem 5.5. *After possibly relabeling the coordinates q_i and the Casimirs χ_i, the $\ell+2$ functions*

$$\chi_i, 1 \le i \le \ell-1, \quad \textit{and} \quad q_\ell,\, q_{\ell+1},\, q_{\ell+2} \tag{26}$$

form a system of (global) coordinates on the affine space N. The Poisson matrix of the ATP-structure on N takes, in terms of these coordinates, the form

$$\widetilde{\Lambda}_N = \begin{pmatrix} 0 & 0 \\ 0 & \Omega \end{pmatrix}, \quad \textit{where} \quad \Omega = c' \begin{pmatrix} 0 & \dfrac{\partial \chi_\ell}{\partial q_{\ell+2}} & -\dfrac{\partial \chi_\ell}{\partial q_{\ell+1}} \\ -\dfrac{\partial \chi_\ell}{\partial q_{\ell+2}} & 0 & \dfrac{\partial \chi_\ell}{\partial q_\ell} \\ \dfrac{\partial \chi_\ell}{\partial q_{\ell+1}} & -\dfrac{\partial \chi_\ell}{\partial q_\ell} & 0 \end{pmatrix}, \tag{27}$$

for some non-zero constant c'. It has the polynomial χ_ℓ as Casimir, which reduces to the polynomial which defines the singularity, when setting $\chi_j = 0$ for $j = 1, 2, \ldots, \ell-1$.

Proof. The non-Poisson part of this theorem is due to Brieskorn and Slodowy. Before proving the Poisson part of the theorem, namely that the Poisson matrix takes the above form (27), we explain, for the convenience of the reader, the basic notions of singularity theory that are used in their proof, see [12] for details. Let (X_0, x) be the germ of an analytic variety X_0 at the point x. A deformation of (X_0, x) is a pair $(\Phi, \imath)$ where $\Phi : X \to U$ is a flat morphism of varieties, with $\Phi(x) = u$, and where the map $\imath : X_0 \to \Phi^{-1}(u)$ is an isomorphism. Such a deformation is called *semi-universal* if any other deformation of (X_0, x) is isomorphic to a deformation, induced from $(\Phi, \imath)$ by a local change of variables, in a neighborhood of x. The semi-universal deformation of (X_0, x) is unique up to isomorphism. It can be explicitly described in the following case: let $(X_0, 0)$ be a germ of a hypersurface of $\mathbf{C}^d$, which is singular at 0, say X_0 is locally given by $f(z) = 0$. Then the semi-universal deformation of $(X_0, 0)$ is the (germ at the origin of the) map

$$\begin{array}{cccc} \Phi & : & \mathbf{C}^k \times \mathbf{C}^d & \to & \mathbf{C}^k \times \mathbf{C} \\ & & (u, z) & \mapsto & (u, F(u, z)), \end{array} \tag{28}$$

where

$$F(u, z) = f(z) + \sum_{i=1}^{k} g_i(z) u_i, \tag{29}$$

and where the polynomials $1, g_1, g_2, \ldots, g_k$ represent a vector space basis of the Milnor (or Tjurina) algebra

$$\mathcal{M}(f) := \frac{\mathbf{C}[z_1, \ldots, z_d]}{\left(f, \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_d}\right)} = \frac{\mathbf{C}[z_1, \ldots, z_d]}{\left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_d}\right)}, \tag{30}$$

where the last equality is valid whenever f is quasi-homogeneous, which is true in the present context. The dimension $\dim \mathcal{M}(f) = k+1$ is called the *Milnor number* of f.

We can now formulate Brieskorn's result. It says that the map $\chi : N \to \mathbf{C}^\ell$, which is the restriction of the adjoint quotient (20) to the slice N, is a semi-universal deformation of the singular surface $N \cap \mathcal{N}$. More precisely, when the Lie algebra is of the type ADE, then the map

$$\begin{array}{cccc} \Phi & : & \mathbf{C}^{\ell-1} \times \mathbf{C}^3 & \to & \mathbf{C}^{\ell-1} \times \mathbf{C} \\ & & ((\chi_1, \ldots, \chi_{\ell-1}), (q_\ell, q_{\ell+1}, q_{\ell+2})) & \mapsto & ((\chi_1, \ldots, \chi_{\ell-1}), \chi_\ell) \end{array} \tag{31}$$

is the semi-universal deformation of the singular surface $N\cap\mathcal{N}$; for the other types one has to consider Γ-invariant semi-universal deformations, as was shown by Slodowy, see Table 2 and [12]. It is implicit in Brieskorn's statement that $(\chi_1,\dots,\chi_{\ell-1},q_\ell,q_{\ell+1},q_{\ell+2})$ form a system of coordinates on N, which comes from the fact that one can solve the $\ell-1$ equations $\chi_i=\chi_i(q)$ *linearly* for $\ell-1$ of the variables q_i, namely the Casimirs have the form

$$\begin{pmatrix}\chi_1\\ \vdots\\ \chi_{\ell-1}\end{pmatrix}=A\begin{pmatrix}q_1\\ \vdots\\ q_{\ell-1}\end{pmatrix}+\begin{pmatrix}F_1(q_\ell,q_{\ell+1},q_{\ell+2})\\ \vdots\\ F_{\ell-1}(q_\ell,q_{\ell+1},q_{\ell+2})\end{pmatrix}, \tag{32}$$

where A is a constant matrix, with $\det A\neq 0$; this will be illustrated in the examples below.

We now get to the Poisson part of the proof. Since the coordinate functions $\chi_1,\dots,\chi_{\ell-1}$ are Casimirs, the Poisson matrix $\widetilde{\Lambda}_N$ takes with respect to these coordinates the block form

$$\widetilde{\Lambda}_N=\begin{pmatrix}0&0\\0&\Omega\end{pmatrix}$$

where Ω is a 3×3 skew-symmetric matrix. We know from Theorem 5.4 that the ATP-structure is a constant multiple of the determinantal structure. Since $\det A\in\mathbf{C}^*$ it follows from (32) that for $\ell\leq i,j\leq \ell+2$,

$$\widetilde{\Lambda}_{ij}:=c\det(\nabla q_i\ \nabla q_j\ \nabla\chi_1\ \dots\ \nabla\chi_\ell)=c'\det(\nabla' q_i\ \nabla' q_j\ \nabla'\chi_\ell),$$

where c and c' are non-zero constants and ∇' denotes the restriction of ∇ to $\mathbf{C}^3$, namely

$$\nabla' F=\begin{pmatrix}\frac{\partial F}{\partial q_\ell} & \frac{\partial F}{\partial q_{\ell+1}} & \frac{\partial F}{\partial q_{\ell+2}}\end{pmatrix}^\top .$$

The explicit formula (27) for Ω follows from it at once. $\square$

Examples

1) For the subregular orbit of $\mathfrak{g}_2$ we have, according to (21), that $\chi_1=q_1$, so that χ_2, expressed in terms of q_2, q_3, q_4 and χ_1, is given by

$$\chi_2=9q_4^2-4q_2^3-4q_3^3+12\chi_1q_2q_3.$$

The Poisson matrix (14) of the ATP-structure is already in the form (27), with $c'=-1/6$ (and $\chi_1=q_1$). Since the Milnor algebra (30) is given in this case by $\mathcal{M}(9q_4^2-4q_2^3-4q_3^3)=\mathbf{C}[q_2,q_3,q_4]/\left(q_2^2,q_3^2,q_4\right)$, one easily sees that 1 and the coefficient q_2q_3 of u_1 form indeed a vector space basis for the Γ-invariant elements of the Milnor algebra (see Table 3); cfr. [12, p. 136].

2) We now turn to the case of $\mathfrak{so}_8$. Recall from (22) that its ATP structure has Casimirs $\chi_1,\dots,\chi_4$. As stated in the proof of Theorem 5.5, we can solve three of them linearly for q_1, q_2, q_3 in terms of χ_1, χ_2, χ_3 and the last three variables q_4, q_5, and q_6. We obtain

$$\begin{aligned}
q_1 &= -q_4-\frac{\chi_1}{2},\\
q_2 &= \frac{1}{64}\left(\chi_1^2-16\chi_3-4\chi_2-32q_4q_5\right),\\
q_3 &= \frac{1}{64}\left(\chi_1^2+48\chi_3-4\chi_2+32q_4q_5\right).
\end{aligned}$$

Substituted in χ_4 this yields

$$\chi_4 = 8q_4q_5^2 - 16q_4^2q_5 - 4q_6^2 - 4\chi_1q_4q_5 + (\chi_2 - \frac{1}{4}\chi_1^2 + 4\chi_3)q_5 - 16\chi_3q_4 - 2\chi_1\chi_3,$$

so that

$$\hat{\chi}_4 = 8q_4q_5^2 - 16q_4^2q_5 - 4q_6^2 - 4\chi_1q_4q_5 + \hat{\chi}_2q_5 - 16\chi_3q_4,$$

where $\hat{\chi}_2 := \chi_2 - \frac{1}{4}\chi_1^2 + 4\chi_3$ and $\hat{\chi}_4 := \chi_4 + 2\chi_1\chi_3$, which can be used instead of χ_2 and χ_4 as basic Ad-invariant polynomials, restricted to N. Using (18), expressed in terms of the coordinates $\chi_1, \hat{\chi}_2, \chi_3, q_4, q_5$ and q_6 we find that the matrix Ω is indeed of the form (27), with $c' = -1/8$, since

$$\begin{aligned}
\{q_4, q_5\} &= q_6 = -\frac{1}{8}\frac{\partial\hat{\chi}_4}{\partial q_6}, \\
\{q_4, q_6\} &= 2q_4q_5 - 2q_4^2 - \tfrac{1}{2}\chi_1q_4 + \tfrac{1}{8}\hat{\chi}_2 = \frac{1}{8}\frac{\partial\hat{\chi}_4}{\partial q_5}, \\
\{q_5, q_6\} &= -q_5^2 + 4q_4q_5 + \tfrac{1}{2}\chi_1q_5 + 2\chi_3 = -\frac{1}{8}\frac{\partial\hat{\chi}_4}{\partial q_4}.
\end{aligned}$$

It follows easily from these formulas that the Milnor algebra is given by

$$\mathcal{M}(8q_4q_5^2 - 16q_4^2q_5 - 4q_6^2) = \mathbf{C}[q_4, q_5, q_6]/(q_6, q_4(q_5 - q_4), q_5(q_5 - 4q_4)),$$

so that 1 and the coefficients q_4, q_5 and q_4q_5 of $\hat{\chi}_4$ form indeed a vector space basis for it.

3) We finally consider the subregular orbit $\mathcal{O}_{sr}$ in $\mathfrak{sl}_4$. This example is from [7]. This case was also examined in [10] where it was shown that the slice, originally due to Arnold [2], belongs to the set $\mathcal{N}_h$. It is the orbit of the nilpotent element

$$e = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The transverse slice in Arnold's coordinates consists of matrices of the form

$$Q = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ q_1 & q_2 & q_3 & q_4 \\ q_5 & 0 & 0 & -q_3 \end{pmatrix}.$$

The basic Casimirs of the ATP-structure, as computed from the characteristic polynomial of Q, are

$$\begin{aligned}
\chi_1 &= q_2 + q_3^2, \\
\chi_2 &= q_1 + q_2q_3, \\
\chi_3 &= q_1q_3 + q_4q_5.
\end{aligned}$$

If we solve the first two equations for the variables q_1, q_2 in terms of χ_1, χ_2 and q_3, q_4, q_5, and substitute the result in χ_3, then we find that

$$\chi_3 = q_3^4 + q_4q_5 - \chi_1q_3^2 + \chi_2q_3.$$

Using the explicit formulas for the ATP-structure, given in [7], expressed in terms of the coordinates χ_1, χ_2, q_3, q_4 and q_5, we find that the matrix Ω is indeed of the form (27), with $c' = 1$, since

$$\begin{aligned}
\{q_3, q_4\} &= q_4 = \frac{\partial \chi_3}{\partial q_5}, \\
\{q_3, q_5\} &= -q_5 = -\frac{\partial \chi_3}{\partial q_4}, \\
\{q_4, q_5\} &= 4q_3^3 - 2\chi_1 q_3 + \chi_2 = \frac{\partial \chi_3}{\partial q_3}.
\end{aligned}$$

It can be read off from these formulas that the Milnor algebra is given by

$$\mathcal{M}(q_3^4 + q_4 q_5) = \mathbf{C}[q_3, q_4, q_5]/(q_4, q_5, q_3^3),$$

so that the coefficients $1, q_3$ and q_3^2 of χ_3 form indeed a vector space basis for it.

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