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INTEGRABLE SYSTEMS AND MODULI SPACES OF RANK 2 VECTOR BUNDLES ON A NON-HYPERELLIPTIC GENUS 3 CURVE

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1. Moduli spaces of rank 2 vector bundles on a Riemann surface of genus 3.

Let Γ be a compact Riemann surface of genus $g > 0$. The simplest non-trivial moduli space that is associated with Γ is $\mathrm{Pic}^d(\Gamma)$, the moduli space of rank 1 vector bundles (line bundles) on Γ of degree d. When $d = 0$ we also speak of the Jacobian of Γ, denoted $\mathrm{Jac}(\Gamma)$; each of the $\mathrm{Pic}^d(\Gamma)$ is isomorphic to $\mathrm{Jac}(\Gamma)$, but not in a canonical way. From the point of view of complex geometry, $\mathrm{Pic}^d(\Gamma)$ is a rather simple object: $\mathrm{Pic}^d(\Gamma)$ is a complex torus $\mathbb{C}^g/\Lambda$, where Λ is a lattice in $\mathbb{C}^g$; notice that its dimension is the genus g of Γ. Since a complex torus is essentially a linear object, in fact a commutative group that is locally isomorphic to $\mathbb{C}^g$, one usually thinks of $\mathrm{Pic}^d(\Gamma)$ as the linearization/abelianization of Γ. From the algebraic point of view, $\mathrm{Pic}^d(\Gamma)$ is a projective variety whose ideal is generated by quadratic polynomials. It is a priori not clear how explicit formulas for these quadratic polynomials can be found: it is only recently that quadratic equations have been obtained for certain two-dimensional complex tori, and this by using techniques that were developed by Adler and van Moerbeke (see [1] and [2]), and that will be explained (and used) later in this article.

The next moduli spaces of interest on Γ are the moduli spaces of rank two bundles on Γ. Let $\mathcal{M}_0(\Gamma)$ denote the moduli space of rank two bundles on Γ with trivial determinant. $\mathcal{M}_0(\Gamma)$ is a smooth variety that compactifies

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naturally into a singular (in general) projective variety, which we denote by $\mathcal{M}(\Gamma)$ (see [6]). This moduli space is closely related to the Kummer variety $\mathrm{Kum}(\Gamma)$, which is the quotient $\mathrm{Jac}(\Gamma)/\sigma$, where σ denotes the reflection with respect to its origin, $\sigma(\xi)=\xi^{-1}$, where $\xi\in\mathrm{Pic}^0(\Gamma)=\mathrm{Jac}(\Gamma)$. Namely, the Kummer variety and the moduli space are naturally embedded in the same projective space $\mathbb{P}^{2^g-1}$, and this in a way which is compatible with the natural embedding of $\mathrm{Kum}(\Gamma)$ in $\mathcal{M}(\Gamma)$, i.e., the following diagram is commutative:

$$\begin{array}{ccc}
\mathrm{Kum}(\Gamma) & \xrightarrow{\quad\imath\quad} & \mathcal{M}(\Gamma)\\
{\scriptstyle\varphi_{\mathcal{L}}}\downarrow & & \downarrow{\scriptstyle\jmath}\\
\mathbb{P}H^0(\mathrm{Jac}(\Gamma),\mathcal{L})^* & \xrightarrow{\ W\ } & \mathbb{P}H^0\bigl(\mathrm{Pic}^{g-1}(\Gamma),[2\Theta]\bigr).
\end{array}$$

Let us explain the different elements that appear in this diagram.

- If we write elements of $\mathrm{Kum}(\Gamma)$ as unordered pairs $\langle\xi,\xi^{-1}\rangle$ with $\xi\in\mathrm{Jac}(\Gamma)$, then $\imath$ is defined by

$$\imath(\langle\xi,\xi^{-1}\rangle)=\xi\oplus\xi^{-1},$$

which is a semi-stable rank 2 bundle on Γ with trivial determinant (it is not a stable bundle, though).

- Θ is the canonical theta divisor on $\mathrm{Pic}^{g-1}(\Gamma)$; it consists of those line bundles on Γ of degree $g-1$ that admit a non-trivial section.
- $[2\Theta]$ is the line bundle that corresponds to (twice) this divisor.
- For $E\in\mathcal{M}(\Gamma)$ we define

$$\mathcal{D}_E:=\bigl\{\xi\in\mathrm{Pic}^{g-1}(\Gamma)\mid\xi\otimes E\text{ has sections}\bigr\}.$$

It can be shown that $\mathcal{D}_E$ is the support of a divisor linearly equivalent to 2Θ, so that we can associate to $\mathcal{D}_E$ (and hence to E) an element of $\mathbb{P}H^0(\mathrm{Pic}^{g-1}(\Gamma),[2\Theta])$. This yields the map $\jmath$, which is an embedding. On $\mathrm{Jac}(\Gamma)$ there is no canonical (theta) divisor, but there is a canonical line bundle $\mathcal{L}$ which gives twice the principal polarization. Consider any $\xi\in\mathrm{Pic}^{g-1}(\Gamma)$ and translate Θ to $\mathrm{Jac}(\Gamma)$ by ξ as well as by $\mathcal{K}_\Gamma\otimes\xi^{-1}$ and take their sum. This gives a divisor on $\mathrm{Jac}(\Gamma)$ which is not canonical, since we chose ξ, but it only depends on the image of ξ in the Kummer variety (of $\mathrm{Pic}^{g-1}(\Gamma)$), hence the rational equivalence class of this divisor is independent of the choice. This yields the above canonical line bundle $\mathcal{L}$.

Then the map $\varphi_{\mathcal{L}}$ is the embedding of $\mathrm{Kum}(\Gamma)$ into $\mathbb{P}H^0(\mathrm{Jac}(\Gamma),\mathcal{L})^*$ that is induced by the canonical map $\mathrm{Jac}(\Gamma)\to\mathbb{P}H^0(\mathrm{Jac}(\Gamma),\mathcal{L})^*$. The projective transformation W, which makes the diagram commute, is sometimes referred to as Wirtinger duality; for its construction we refer to [9].

Narasimhan and Ramanan proved in 1969 (see [8]) the following theorem.

Theorem 1.1 (Narasimhan-Ramanan). — *If Γ is a compact Riemann surface of genus at least* 3 *then* $\mathrm{Kum}(\Gamma)$ *is the singular locus of* $\mathcal{M}(\Gamma)$.

In this theorem, the singular varieties $\mathrm{Kum}(\Gamma)$ and $\mathcal{M}(\Gamma)$ are both viewed as living in $\mathbb{P}^{2^g-1}$, via the embeddings $\imath$ and $\jmath$. Thus, every semi-stable rank 2 bundle on Γ that is not stable is of the form $\xi\oplus\xi^{-1}$, where ξ is a line bundle of degree zero on Γ. The case $g=2$ is exceptional because the moduli space $\mathcal{M}(\Gamma)$ is $\mathbb{P}^3$, so it is non-singular (in a *certain* sense, however, it *is* singular along $\mathrm{Kum}(\Gamma)$, see [7]).

Fifteen years later, Narasimhan and Ramanan proved the following related result (see [9]).

Theorem 1.2 (Narasimhan-Ramanan). — *If Γ is a compact Riemann surface of genus* 3 *and Γ is non-hyperelliptic then $\mathcal{M}(\Gamma)$ is a quartic hypersurface of $\mathbb{P}^7$.*

Recall that a Riemann surface is non-hyperelliptic if and only if the canonical map $\varphi_{\mathcal{K}_\Gamma}:\Gamma\to\mathbb{P}H^0(\Gamma,\mathcal{K}_\Gamma)^*$ is an embedding; a generic compact Riemann surface of genus 3 is non-hyperelliptic. For the case of rank 2 vector bundles on hyperelliptic Riemann surfaces (of genus g), where the moduli space can be explicitly described as a variety of linear subspaces of $\mathbb{P}^{2g+1}$, see [5].

Denoting by Q the quartic polynomial that defines $\mathcal{M}(\Gamma)$ (as a quartic in $\mathbb{P}^7$) it is a simple consequence of these two theorems that $\mathrm{Kum}(\Gamma)$ is given as the intersection of eight cubic hypersurfaces, namely the cubics $\partial Q/\partial x_i=0$, for $i=0,\dots,7$, where $x_0,\dots,x_7$ are any projective coordinates on $\mathbb{P}^7$.

The purpose of this paper is to compute an explicit equation of this quartic hypersurface for a family of non-hyperelliptic Riemann surfaces of genus 3, and this by using the theory of integrable systems. Here, "explicit" means that the coefficients of the quartic are explicit polynomials in the coefficients that appear in an algebraic equation of the Riemann surface as a plane algebraic curve. Our technique is to first construct an algebraic completely integrable systems whose generic fiber of the momentum map is an affine part of the Jacobian of a non-hyperelliptic Riemann surface Γ of genus 3. Then we construct an embedding of the Kummer variety $\mathrm{Kum}(\Gamma)$ in $\mathbb{P}^7$ by using the sections of $\mathcal{L}$ and we compute the (eight-dimensional) vector space of all cubic polynomials that vanish on the image. In view of Wirtinger duality we may think of these cubic polynomials as being defined on $\mathbb{P}H^0(\mathrm{Pic}^{g-1}(\Gamma),[2\Theta])$, where they define the singular locus of the moduli space $\mathcal{M}(\Gamma)$. The polynomial Q that defines $\mathcal{M}(\Gamma)$ is then found by a simple integration procedure. Our equation, which is valid for a whole family of curves, is easily specialized to particular curves; having an equation for a whole family of moduli spaces is not just interesting from the point of view of deformation theory, but it indispensable for possible applications to the Knizhnik-Zamolodchikov equation.

2. Construction of the integrable system.

In this section we construct an integrable system whose invariant manifolds are affine parts of Jacobians of non-hyperelliptic Riemann surfaces of genus 3; see [10], Chapter VI for more details and generalizations. We consider the space $\mathcal{N}$ of pairs (P,Q), where

- P,Q formal differential operators (in $\partial=\partial/\partial x$),
- $\mathrm{ord}\, Q=3$ and $\mathrm{ord}\, P=4$,
- P monic, $P=\partial^4+\mathcal{O}(\partial^3)$,
- Q normalized, $Q=\partial^3+\mathcal{O}(\partial^1)$,
- $[P,Q]=0$.

There is a natural matrix, associated to such a commuting pair. To construct it, we need to Sato Grassmannian, whose definition we recall shortly. Let $\Psi=\mathbb{C}[[x]]((\partial^{-1}))$ denote the algebra of formal pseudo-differential operators and let **Volt** denote the group of monic, zeroth order elements of Ψ, called the *Volterra group.* Let δ denote Dirac's delta function, thought of as a zeroth order differential operator. It has the fundamental property that for any $Q\in\Psi$ there exists a unique $Q^c\in\Psi$ with constant coefficients, such that $Q\delta=Q^c\delta$. The left coset $V:=\mathbb{C}((\partial^{-1}))\delta=\Psi^c\delta$ is a left Ψ-module in a natural way: for $P\in\Psi$ and for $Q\in\mathbb{C}((\partial^{-1}))\subset\Psi$ we define $P\cdot(Q\delta)=(PQ)\delta$. For $Q\in\Psi$ we define $W_Q\subset\mathbb{C}((\partial^{-1}))\delta$ by $W_Q=Q\cdot H$, where $H\subset V$ is defined by $H=\mathbb{C}[\partial]\delta$. The set of all W_T, where T belongs to **Volt** is the Sato Grassmannian, denoted **GR**.

We can now explain the construction. Since Q is normalized and has order q, there exists an element $T\in\mathbf{Volt}$ such that $Q=T^{-1}\partial^qT$. Choosing such an element T we define $W=W_T=T\cdot H\in\mathbf{GR}$. If we let $\widetilde{P}=TPT^{-1}$ then $\widetilde{P}\in\Psi$ is monic of order p and $[\partial^q,\widetilde{P}]=0$, so that $\widetilde{P}$ has constant coefficients. Thus, there corresponds to the pair (P,Q) a pair $(\widetilde{P},W)$, where $W\in\mathbf{GR}$ and where $\widetilde{P}\in\Psi$ has constant coefficients. The pair is unique, up to multiplication by an element of **Volt**, with constant coefficients. The important property is that W is stable under the action of ∂^3 and $\widetilde{P}$, i.e., $\partial^3\cdot W\subset W$ and $\widetilde{P}\cdot W\subset W$. The first inclusion follows from $\partial^q\cdot W=T\cdot(Q\cdot H)\subset T\cdot H=W$,, where we have used that $Q\cdot H\subset H$ holds because Q is a differential operator. The second inclusion is proven in the same way. The first inclusion yields the existence of a periodic basis for W, while the second one leads to a (trace-less) 3×3 matrix $\widetilde{X}$, which is $\widetilde{P}$, written in terms of this periodic basis. The entries of $\widetilde{X}$ are polynomials in $\lambda:=\partial^3$; since P is monic of degree 4, the same is true for $\widetilde{P}$, so that the degrees of the entries of $\widetilde{X}$ have the following degree constraints:(1)

$$\begin{pmatrix}\le 1 & 1 & \le 0\\ \le 1 & \le 1 & 1\\ 2 & \le 1 & \le 1\end{pmatrix}.$$

The matrix $\widetilde{X}$ is not unique, in fact it is only defined up to conjugation by a lower triangular matrix. It is easy to see that this conjugation class contains a unique element of the 10-dimensional affine subspace M of $\mathrm{sl}(3)[\lambda]$, whose elements have the form

$$X(\lambda):=\begin{pmatrix} b_{11} & \lambda+b_{12} & b_{13}\\ b_{21} & b_{22} & \lambda+b_{23}\\ \lambda^2+a_{31}\lambda+b_{31} & a_{32}\lambda+b_{32} & -b_{11}-b_{22}\end{pmatrix}.$$

This yields a well-defined map $\mathcal{N}\to M$, that associates to a commuting pair $(P,Q)\in\mathcal{N}$ an element $X(\lambda)$ of M. In fact, it can be shown that inversely to any element of M one can associate a pair of commmuting differential operator $(P,Q)\in\mathcal{N}$ (see [10], Chap. VI), so that the constructed map is in fact a bijection.

This affine space M is the manifold that underlies our integrable system. To see how it relates to our original problem, consider for $X(\lambda)\in M$

(1) An entry such as "≤ 1" means that the degree of the polynomial is at most 1, while "1" means that the polynomial is monic of degree 1.

its characteristic polynomial $|\mu\,\mathrm{Id}_3 - X(\lambda)|$, and let $H : M \to \mathbb{C}^7$ be the polynomial map, which is defined by its coefficients in λ and μ, say

$$|\mu\,\mathrm{Id}_3 - X(\lambda)| = \mu^3 - \mu(H_1\lambda^2 + H_2\lambda + H_3) - (\lambda^4 + H_4\lambda^3 + H_5\lambda^2 + H_6\lambda + H_7).$$

For any $X(\lambda) \in M$ an affine algebraic curve $\Gamma_h^0 \subset \mathbb{C}^2$ is defined by $|\mu\,\mathrm{Id}_3 - X(\lambda)| = 0$, where $h := H(X(\lambda))$. Thus, the spectral curves that we find here are precisely the type of curves that we are interested in: if $X(\lambda)$ is a generic element of M then Γ_h^0 is a non-singular non-hyperelliptic curve of genus 3. We denote by $\mathcal{H}$ the Zariski open subset of $\mathbb{C}^7$ consisting of those h for which Γ_h^0 is non-singular. For $h \in \mathcal{H}$, the compact Riemann surface that corresponds to Γ_h^0 is denoted by Γ_h. For future use we also introduce the set $\mathcal{H}_0$ of those (α, β, γ) for which the affine curve

$$y^3 = \lambda^4 + \alpha\lambda^2 + \beta\lambda + \gamma \tag{1}$$

is non-singular. Explicitly this means that the parameters α, β and γ are such that

$$27\beta^4 + 4\alpha(\alpha^2 - 36\gamma)\beta^2 - 16\gamma(\alpha^2 - 4\gamma)^2 \neq 0. \tag{2}$$

$\mathcal{H}_0$ is naturally identified with a subset of $\mathcal{H}$.

We now get to the Hamiltonian structure and to the commuting vector fields on M, that will make up the integrable system. On the space of normalized differential operators of a fixed order, such as Q (which has order 3) there is a natural set of commuting vector fields, the KP hierarchy. Explicitly, it is given by $\mathrm{d}Q/\mathrm{d}t_i = [Q_+^{i/q}, Q]$, where $+$ denotes the differential part of a pseudo-differential operator. These vector fields induce commuting vector fields on $\mathcal{N}$, by putting

$$\frac{\mathrm{d}Q}{\mathrm{d}t_i} = [Q_+^{i/q}, Q], \qquad \frac{\mathrm{d}P}{\mathrm{d}t_i} = [Q_+^{i/q}, P].$$

Under the bijection $\mathcal{N} \leftrightarrow M$ these vector fields correspond to commuting vector fields on M, where the simplest one is given by the Lax equation $\dot{X}(\lambda) = [X(\lambda), Y(\lambda)]$, where

$$Y_1(\lambda) := \begin{pmatrix} 0 & 1 & 0 \\ -b_{13} & 0 & 1 \\ \lambda + a_{31} - b_{23} & a_{32} - b_{13} & 0 \end{pmatrix}.$$

To write the commuting vector fields, we first point out that the matrix $Y_1(\lambda)$ is of the form $[X(\lambda)/\lambda]_+$ plus a strictly lower triangular matrix, where the index $+$ means now hat we take the polynomial part (in λ). Then $\mathcal{V}_i$, for $i = 2, 3$ are of the form $\dot{X}(\lambda) = [X(\lambda), Y_i(\lambda)]$, where

$$Y_i(\lambda) = \left[\frac{A^2(\lambda)}{\lambda^{i-1}}\right]_+ + \begin{pmatrix} 0 & 0 & 0 \\ u_i & 0 & 0 \\ v_i & u_i & 0 \end{pmatrix},$$

with $u_2 = -b_{12} - b_{23}$ and $v_2 = b_{11}$ on the one hand, and $u_3 = b_{13}b_{22} - b_{12}b_{23}$ and $v_3 = b_{11}b_{23} - b_{21}b_{13}$ on the other hand. Moreover, these three vector fields are Hamiltonian with respect to a Poisson bracket $\{\cdot\,,\cdot\}$ on M which is a reduction of the standard R-bracket that comes from the splitting of the affine Lie algebra $\mathrm{sl}(3)[\lambda, \lambda^{-1}]$ into polynomials in λ and polynomials in λ^{-1} without constant term. For the details of this construction, we refer to [10, Chap. VI].

The main characteristics of this integrable system are summarized in the following theorem.

Theorem 2.1. — *$(M, \{\cdot\,,\cdot\}, H)$ is an algebraic completely integrable system: it is integrable in the sense of Liouville and moreover, if $h \in \mathcal{H}$ then the fiber $H^{-1}(h)$ is isomorphic to $\mathrm{Jac}(\Gamma_h)$ minus a divisor $\mathcal{D}_h$, which is a translate of the theta divisor Θ_h, and the Hamiltonian vector fields $\{\cdot\,, H_i\}$ are linear on $\mathrm{Jac}(\Gamma_h)$.*

The proof of the above theorem follows from the fact that the map which assigns to a matrix $X(\lambda) \in M$ the projectivized eigenvector map $X(\lambda)$ is injective; the proof of this injectivity is an essential ingredient in establishing the bijection between M and the above space of pairs of differential operators (see [10, Chap. VI]).

3. Embedding Kum(Γ) in $\mathbb{P}^7$.

For fixed $h \in \mathcal{H}$ we use the techniques developed by Adler and van Moerbeke (see [1] and [2, Chap. VII]) to compute explicitly a basis for the 8-dimensional vector space of holomorphic functions on $H^{-1}(h) \cong \mathrm{Jac}(\Gamma_h) \setminus \mathcal{D}_h$ which have a double pole at most when extended to meromorphic functions on $\mathrm{Jac}(\Gamma_h)$. To do this we search for the family of Laurent solutions to $\mathcal{V}_1$ which depends on $\dim M - 1 = 9$ free parameters (this is also called the principal balance); there exists precisely one such balance because the divisor $\mathcal{D}_h$ is a translate of the theta divisor, in particular it is irreducible. In the present case this balance turns out to be weight homogeneous, hence it can be found algorithmically. In fact, if we assign weights to the phase variables according to Table 1,

1	2	3	4	5	6
	b_{13}	b_{12}	b_{11}	b_{21}	b_{31}
	a_{32}	b_{23}	b_{22}	b_{32}	
		a_{31}			

Table 1. The weights of the phase variables

then we find that the weights of the constants of motion H_i are given by

$$\varpi(H_1,\dots,H_7) = (2,5,8,3,6,9,12),$$

and that $\mathcal{V}_1$ is a weight homogeneous vector field (which means that $\mathcal{V}_1$ has weight 1). Using these weights one computes algorithmically all weight homogeneous Laurent solutions to $\mathcal{V}_1$ by substituting for each of the phase variables x the first $k+1$ terms of a general Laurent polynomial that starts at $t^{-\varpi(x)}$, where $\varpi(x)$ denotes the weight of x. For $k=0$ this leads to a non-linear system of equations, called the indicial equation, which admits in the present case the solutions that are given in Table 2.

a_{31}	a_{32}	b_{11}	b_{12}	b_{13}	b_{21}	b_{22}	b_{23}	b_{31}	b_{32}
-4	2	-4	4	-2	0	0	0	8	-8
4	2	8	0	-2	16	0	-4	32	0
0	5	20	5	-5	80	-15	-5	200	-55
0	1	0	1	-1	0	1	-1	0	1

Table 2. The four solutions to the indicial equation

After the zeroth step one only gets linear equations, which are governed by the Kowalevski matrix $\mathcal{K}$, defined by

$$\mathcal{K}_{ij} := \frac{\partial f_i}{\partial x_j} + \varpi(x_i)\delta_{ij}, \qquad 1 \le i,j \le 10,$$

where $x_1,\dots,x_{10}$ are the phase variables, taken for example in the order which is given in Table 2. Since ℓ free parameters can only appear at those steps k for which k is an eigenvalue with multiplicity ℓ of $\mathcal{K}$ we compute for

the four points given in Table 2 the characteristic polynomial $|\mu\mathcal{I}_{10} - \mathcal{K}|$ of $\mathcal{K}$ and we find, in that order,

$$(\mu+2)(\mu+1)(\mu-2)(\mu-3)^2(\mu-4)(\mu-5)(\mu-6)(\mu-8)(\mu-9),$$
$$(\mu+2)(\mu+1)(\mu-2)(\mu-3)^2(\mu-4)(\mu-5)(\mu-6)(\mu-8)(\mu-9),$$
$$(\mu+5)(\mu+2)(\mu+1)(\mu-2)(\mu-3)(\mu-5)(\mu-6)(\mu-8)(\mu-9)(\mu-12),$$
$$(\mu+1)(\mu-1)(\mu-2)(\mu-3)^2(\mu-4)(\mu-5)(\mu-6)^2(\mu-8).$$

It follows that only the last solution in Table 2 can lead to the principal balance. By direct computation one finds that this solution leads indeed to a family of Laurent solutions depending on 9 free parameters. The first few terms are given by

$$\begin{aligned}
a_{31}(t) &= \frac{2a}{t^2} + d + O(t),\\
a_{32}(t) &= \frac{1}{t^2} + 5a^2 + 2b + (2ab - d - 2c)t + O(t^2),\\
b_{11}(t) &= \frac{2a}{t^3} + \frac{2a^2}{t^2} + \frac{4ab - 2d - 4c}{t} + O(t^0),\\
b_{12}(t) &= \frac{1}{t^3} - \frac{a}{t^2} + d + c - 2ab + O(t),\\
b_{13}(t) &= -\frac{1}{t^2} + a^2 + b + (d + 2c - 2ab)t + O(t^2),\\
b_{21}(t) &= \frac{2a}{t^4} + 4\frac{a^2}{t^2} + \frac{6ab + 4a^3 - 2d - 4c}{t^2} + O(t^{-1}),\\
b_{22}(t) &= \frac{1}{t^4} + \frac{b}{t^2} + \frac{2d + 4c - 4ab}{t} + O(t^0),\\
b_{23}(t) &= -\frac{1}{t^3} - \frac{a}{t^2} - c + O(t),\\
b_{31}(t) &= \frac{2a}{t^5} + \frac{6a^2}{t^4} + \frac{8ab + 10a^3 - 2d - 4c}{t^3} + O(t^{-2}),\\
b_{32}(t) &= \frac{1}{t^5} + \frac{a}{t^4} + \frac{a^2 + 2b}{t^3} + \frac{2d + 3c - 4ab - a^3}{t^2} + O(t^{-1}),
\end{aligned}$$

where $a, b, \dots$ are the free parameters. A few more terms are needed to do the computations that follow, but they are easily computed (using a computer) from the given terms.

We now proceed to compute a basis for the 8-dimensional vector space of regular functions (polynomials) on $H^{-1}(h)$ which have the property that, when they are viewed as meromorphic functions on $\mathrm{Jac}(\Gamma_h)$ then they have a double pole at worst at $\mathcal{D}_h$. This can be done by using the above Laurent solutions in view of the following theorem ([2], Prop. 6.14, specialized to our case).

Theorem 3.1. — *Let P be a polynomial in the phase variables and let $h \in \mathcal{H}$. The pole order of $P_{|H^{-1}(h)}$, viewed as a meromorphic function on $\mathrm{Jac}(\Gamma_h)$, along $\mathcal{D}_h$ is equal to the pole order (in t) of the Laurent series $P(t)$, obtained by substituting the first few terms of the principal balance in P.*

It is easy to see that if P is a polynomial in the phase variables, such that $P(t)$ has a pole order p, and $P = P_0 + \cdots + P_s$, where P_i consists of the terms of P that have weight i, then each P_i has a pole order smaller than or equal to p. This implies that it suffices to search for weight homogeneous polynomials P for which $P(t)$ has a pole of order 2 at most. We arrive in this case at the following list of eight weight homogeneous polynomials.

$$
\begin{aligned}
z_0 &:= 1,\\
z_1 &:= a_{32},\\
z_2 &:= a_{31},\\
z_3 &:= b_{22} - a_{32}^2,\\
z_4 &:= b_{22}b_{13} - b_{12}b_{23},\\
z_5 &:= b_{13}(a_{32}b_{23} + a_{31}b_{13}) + b_{11}b_{23} - b_{12}(b_{11} + b_{22}),\\
z_6 &:= b_{13}(b_{12}b_{23} - b_{22}b_{13} + b_{31}) + b_{11}(b_{11} + b_{22}),\\
z_7 &:= (b_{11} - b_{22})a_{32}^3 - \big(a_{31}(b_{12} + b_{23}) + b_{12}b_{23} + b_{31}\big)a_{32}^2\\
&\qquad - \big(a_{31}(b_{21} + b_{32}) - b_{23}b_{32} - 2b_{22}^2\big)a_{32} + a_{31}b_{12}(b_{11} + 2b_{22})\\
&\qquad\qquad + a_{31}b_{23}(b_{22} - b_{11}) + b_{12}b_{22}b_{23} + b_{21}b_{32}.
\end{aligned}
$$

Their weights are $\varpi(z_0, \ldots, z_7) = (0, 2, 3, 4, 6, 7, 8, 10)$.

By computing the leading terms in the series $z_i(t)$ (which is a rational function on the translate $\mathcal{D}_h$ of the theta divisor) one shows easily that these functions are indeed linearly independent, when restricted to $H^{-1}(h)$. Therefore the closure of the image of the canonical map

$$
\varphi : H^{-1}(h) \longrightarrow \mathbb{P}^7, \quad p \longmapsto \big(1 : z_1(p) : \cdots : z_7(p)\big) \tag{3}
$$

is $\mathrm{Kum}(\Gamma_h)$, embedded in $\mathbb{P}H^0(\mathrm{Jac}(\Gamma_h), 2\mathcal{D}_h)^*$.

4. An equation for the moduli space.

The next step is to determine a basis of the (eight-dimensional) vector space of homogeneous cubic polynomials that vanish on the image of the map φ that we have constructed. The following considerations are extremely helpful for doing this. We know that there exists a homogeneous quartic polynomial $Q(z_0, \ldots, z_7)$ which yields eight linearly independent cubic polynomials, vanishing on the image of φ, by differentiating Q which respect to each of the z_i. The coefficients of this polynomial are functions of the values h_j of the constants of motion H_j. Weight homogeneity of the z_i and the H_j (with respect of the weights for the phase variables that were given in Table 2) implies that the rescaling map, which amounts to multiplying each of the z_i by $\nu^{\varpi(z_i)}$ and h_j by $\nu^{\varpi(H_j)}$ yields the same quartic polynomial Q, up to a constant, which implies that Q is weight homogeneous (taking into account the weights of the h_i), and in particular that the coefficients are polynomial functions in the h_i. Moreover, we can determine the weight of Q as soon as we know the cubic polynomial of lowest weight that vanishes on the image of φ, since that one is necessarily $\partial Q/\partial z_7$, and since $\varpi(Q) = \varpi(\partial Q/\partial z_i) + \varpi(z_i)$; the latter formula then allows us to determine the weights of the other cubics.

In order to find a cubic polynomial C of a given weight d that vanishes on the image of φ, write

$$
C = \sum_{i \le j \le k = 0}^{7} C_{ijk}(H) z_i z_j z_k,
$$

where each C_{ijk} is the most general polynomial in $H_1 \ldots, H_7$, which is weight homogeneous of weight $d - \varpi(z_i) - \varpi(z_j) - \varpi(z_k)$. Then substitute the definitions of the z_i and the H_j in terms of the phase variables and express that the resulting polynomial in the phase variables is identically zero, which gives a huge system of linear equations on all the coefficients that appear in the polynomials C_{ijk}. The non-trivial polynomial of lowest weight that is found has weight 14 and is given (up to a constant) by

$$
\begin{aligned}
C_1 := 2(z_7 - \alpha z_3)z_1^2 &+ (2\gamma z_0^2 + (\beta z_2 - 2\alpha z_4)z_0 + 2z_4^2 - z_3 z_6)z_1\\
&+ (z_5^2 - 4z_4 z_6)z_0 - z_2 z_3 z_5 - z_2^2 z_6 - z_3^2 z_4,
\end{aligned}
$$

where we have taken $h = (\alpha, \beta, \gamma) \in \mathcal{H}_0$, the full C_1 for $h \in \mathcal{H}$ being three times as long (but it has the same degree and contains, among others, the above terms). It follows that Q has weight $\varpi(Q) = \varpi(C_1) + \varpi(z_7) = 24$, and Q can be found by taking the most general quartic polynomial in $z_0, \dots, z_7$, whose coefficients are polynomials in $H_1, \dots, H_7$, and expressing that each of its first order derivatives vanishes on the image of φ. In practice this amounts to solving more than 1000 linear equations in a comparable number of unknowns, which most (all?) current computer programs (on a PC) have trouble with. There is however a less costly procedure, that we describe now. First one computes a basis $C_1, \dots, C_8$ for the cubic polynomials that vanish on the image of φ, using the method that is described above. We know that such a basis can be constructed with cubic polynomials of weight $\varpi(C_i) = 24 - \varpi(z_i)$, which amounts to

$$\varpi(C_1, \dots, C_8) = (14, 16, 17, 18, 20, 21, 22, 24).$$

This has to be done somehow carefully, because we want these cubic polynomials to be linearly independent, when restricted to particular values h of H. Indeed, if we multiply C_1 by H_1 then we find a cubic polynomial of weight 16 that vanishes on the image of φ, but its restriction to $H = h$ is just a multiple of C_1. Besides this fact the computation of these cubic polynomials is long but straightforward.

Having found these cubic polynomials $C_1, \dots, C_8$ it is because of the choices involved in their construction unlikely that they will exist a polynomial Q such that $\partial Q/\partial z_{8-i} = C_i$, for $i = 1, \dots, 8$; in fact this will only be the case if the integrability property $\partial C_i/\partial z_{8-j} = \partial C_j/\partial z_{8-i}$ holds for $1 \le i, j \le 8$. However, we know that some other basis of $\operatorname{Span}\{C_1, \dots, C_8\}$ must satisfy this property. To find this basis, let $R_{ij} := \partial C_i/\partial z_{8-j}$ and let A be an 8×8 matrix. Then the elements of AC satisfy the mentioned integrability property if and only if AR is symmetric. Thus, by simple linear algebra we determine a basis $(C_1', \dots, C_8')$ of the cubic polynomials which are derivatives of the quartic polynomial Q.

Below, we state the result for a particular (3-dimensional) family of curves Γ_h (namely $h \in \mathcal{H}_0$), since the formula for the whole seven-dimensional space (where $h \in \mathcal{H}$) is much longer; the reader will have no difficulty computing it, using the same methods.

THEOREM 4.1. — *Let $(\alpha,\beta,\gamma) \in \mathcal{H}_0$ so that the algebraic curve*

$$y^3 = \lambda^4 + \alpha\lambda^2 + \beta\lambda + \gamma$$

is smooth, and let Γ_h denote its compact Riemann surface. The moduli space $\mathcal{M}(\Gamma_h)$ is given by the following quartic hypersurface in $\mathbb{P}^7$,

$$\gamma^2 z_0^4 + P_1 z_0^3 + P_2 z_0^2 + P_3 z_0 + P_4 = 0, \tag{4}$$

where

$$\begin{aligned}
P_1 &= (2\alpha\gamma - \beta^2)z_4 + \beta\gamma z_2,\\
P_2 &= (2\gamma z_7 + (2\alpha\gamma - \beta^2)z_3)z_1 + \beta z_5 z_6 + \alpha\gamma z_2^2\\
&\qquad - \alpha\beta z_2 z_4 + 3\gamma z_3 z_6 + (\alpha^2 + 2\gamma)z_4^2,\\
P_3 &= (4\gamma\alpha - \beta^2)z_1^3 + 4\gamma z_6 z_1^2 + (\beta z_2 z_7 - 2\alpha z_4 z_7 - \alpha\beta z_2 z_3\\
&\qquad - 2\beta z_4 z_5 + 4\gamma z_2 z_5 + 2\alpha^2 z_3 z_4 - \alpha z_6^2)z_1 + z_5^2 z_7\\
&\qquad + \alpha z_2 z_5 z_6 - z_6^3 + 2\alpha z_4^3 + 3\alpha z_3 z_4 z_6 - 4 z_4 z_6 z_7\\
&\qquad + \gamma z_3^3 + \beta z_3^2 z_5 - 3\beta z_2 z_4^2 + \beta z_2 z_3 z_6 + 4\gamma z_2^2 z_4,\\
P_4 &= ((z_7 - \alpha z_3)^2 + \gamma z_3^2 + \alpha z_5^2 + \beta z_3 z_5)z_1^2 + (z_5^2 z_6 - z_3 z_6 z_7\\
&\qquad + \alpha z_3^2 z_6 + \beta z_2^2 z_5 + 2 z_4^2 z_7 - 2\alpha z_2 z_4 z_5 - 2\alpha z_3 z_4^2 + 2\gamma z_2^2 z_3\\
&\qquad - \beta z_2 z_3 z_4)z_1 - z_2 z_3 z_5 z_7 - z_3^2 z_4 z_7 - z_2^2 z_6 z_7 - z_3 z_4^2 z_6\\
&\qquad + \alpha z_2^2 z_3 z_6 - 2 z_2 z_4 z_5 z_6 + \gamma z_2^4 + z_4^4 + \alpha z_2^2 z_4^2 + \alpha z_3^3 z_4\\
&\qquad - \beta z_2^3 z_4 + \alpha z_2 z_3^2 z_5 + z_3 z_4 z_5^2 + z_2 z_5^3.
\end{aligned}$$

Notice that the cubic polynomial Q_1 is indeed given by $C_1 = \partial Q/\partial z_7$. The other basic cubic polynomials are found by computing $\partial Q/\partial z_i$ for $i = 0, \dots, 6$.

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