A special case of the Garnier system, (1,4)-polarized Abelian surfaces and their moduli

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Abstract

In this paper we discuss some algebraic-geometric aspects of a (family of) integrable quartic potential(s) in two degrees of freedom. It is a special case of the so-called Garnier system, which was first introduced by Garnier when studying isomonodromic deformations of differential equations. We show that the complex invariant manifolds of this integrable system complete into Abelian surfaces of type $(1,4)$ and use the specific geometry of these surfaces to prove that the system is algebraic completely integrable. The limiting case of the potential $(q_1^2+q_2^2)^2$ will also be discussed, in particular a Lax pair for this limiting potential will be found from the Lax pair we construct for the generic case.

We also show that every Abelian surface of type $(1,4)$ occurs as an invariant manifold for one of these integrable potentials. This allows us (among other explicit things) to compute explicitely a canonical map between the moduli space of Abelian surfaces of type $(1,4)$ to the moduli space of Jacobians of genus two curves.

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1. Introduction

It is well-known that there is a rich interaction between algebraic geometry and algebraic completely integrable systems (a.c.i. systems) both in the finite-dimensional case (e.g. Toda lattices, geodesic flows on Lie groups, classical tops) and the infinite-dimensional case (e.g. KdV and KP equations, non-linear Schrödinger equation) (see [AvM1], [D], [M2], [Sh]).

The main fact is that the generic integral curve of the Hamiltonian vector field of such an integrable system is dense in an Abelian variety, i.e., in a complex algebraic torus (run with complex time). The different Abelian varieties which correspond to the different integral curves fill up the phase space and are called the (complex) *invariant manifolds* of the vector field. Equations for (an affine part of) these invariant manifolds are given by a maximal set of independent functions, invariant for the vector field (often called *constants of motion* or *first integrals*) one of which is the Hamiltonian function defining the vector field. It follows that knowing these constants of motion leads to explicit equations for affine parts of Abelian surfaces. On the one hand they yield by direct methods some interesting results about the family of Abelian varieties which appear in the system, which often describe the full moduli of Abelian varieties of a given type (at least in small dimensions). Remember that Abelian varieties (of dimension g) are described by means of a set of discrete parameters $(\delta_1, \ldots, \delta_g)$ giving the (polarization) *type* and by means of a Riemann matrix Z (i.e., a symmetric $g \times g$ matrix with positive definite imaginary part). On the other hand algebraic geometry can be used to study the integrable system, for example to linearize the flow of the vector field or to find transformations between different systems (see [V1] and Section 2.2 below).

The present paper deals with an integrable system defined by a quartic potential in two degrees of freedom, whose generic invariant manifolds are Abelian surfaces of polarization type $(1, 4)$. In one direction, the specific geometry of these Abelian surfaces will be used to prove algebraic complete integrability of the potential and in the other direction the explicit (affine) coordinates provided by the system will be used to prove some new results and perform some explicit constructions for Abelian surfaces of type $(1, 4)$. In this way we provide and exploit an essentially new case of the interaction between algebraic geometry and a.c.i. systems (the present potential is the first known a.c.i. system leading to Abelian surfaces of type $(1, 4)$).

The potential is a quadratic perturbation

$$V_{\alpha\beta} = \left(q_1^2 + q_2^2\right)^2 + \alpha q_1^2 + \beta q_2^2 \tag{1}$$

of the potential

$$V_{00} = \left(q_1^2 + q_2^2\right)^2,$$

the latter being obviously integrable since it is a central potential. However, although V_{00} as well as $V_{\alpha\alpha}$ are only Liouville integrable (but not a.c.i.) the perturbation $V_{\alpha\beta}$ becomes a.c.i. for $\alpha \neq \beta$. $V_{\alpha\beta}$ can be interpreted as a potential which describes an anisotropic harmonic oscillator in a central field; remark that the central field V_{00} is exceptional in the sense that an anisotropic harmonic oscillator in a general central field is not integrable.

Newton's equations of motion take the symmetric form

$$\begin{aligned}\ddot{q}_1 &= -2q_1\left(2q_1^2 + 2q_2^2 + \alpha\right),\\ \ddot{q}_2 &= -2q_2\left(2q_1^2 + 2q_2^2 + \beta\right),\end{aligned}$$

and it is checked at once that

$$F=(q_1\dot q_2-q_2\dot q_1)^2-(\beta-\alpha)\left(\dot q_1^2+2q_1^4+2q_1^2q_2^2+2\alpha q_1^2\right)$$

is a constant of motion, independent of the Hamiltonian

$$H=\frac{1}{2}\left(\dot q_1^2+\dot q_2^2\right)+\left(q_1^2+q_2^2\right)^2+\alpha q_1^2+\beta q_2^2\,.$$

It was pointed out to me by A. Perelomov that this potential was first studied by Garnier in the beginning of this century. In fact the Garnier system is a much more general system which contains a lot of integrable systems; the derivation of the potentials $V_{\alpha\beta}$ (and their generalizations to higher dimensions) will be given in the Appendix (see [G], [P]).

To prove that the potentials $V_{\alpha\beta}$ define an a.c.i. system we use the result of [BLS] (explained in Section 2.1) which states that the line bundle $\mathcal{L}$ which defines the polarization on a generic Abelian surface of type $(1,4)$ induces a birational map $\phi_{\mathcal{L}}\colon\mathcal{T}^2\to\mathbb{P}^3$, whose image is an octic of a certain type; an equation for this octic is given with respct to well-chosen coordinates for $\mathbb{P}^3$ by

$$\begin{aligned}&\lambda_0^2y_0^2y_1^2y_2^2y_3^2+\lambda_1^2(y_0^4y_1^4+y_2^4y_3^4)+\lambda_2^2(y_1^4y_3^4+y_0^4y_2^4)+\lambda_3^2(y_0^4y_3^4+y_1^4y_2^4)+\\&2\lambda_1\lambda_2(y_0^2y_1^2+y_2^2y_3^2)(y_1^2y_3^2-y_0^2y_2^2)+2\lambda_1\lambda_3(y_0^2y_3^2-y_1^2y_2^2)(y_0^2y_1^2-y_2^2y_3^2)+\\&2\lambda_2\lambda_3(y_1^2y_2^2+y_0^2y_3^2)(y_1^2y_3^2+y_0^2y_2^2)=0,\end{aligned}\tag{2}$$

for some $(\lambda_0\colon\lambda_1\colon\lambda_2\colon\lambda_3)\in\mathbb{P}^3\setminus S$ where S is some divisor of $\mathbb{P}^3$, which we will determine. Moreover each octic of this type occurs in that way. It will allow us to show that the invariant surfaces of the Hamiltonian vector field associated to the potential $V_{\alpha\beta}$, $(\alpha\neq\beta)$, are Abelian surfaces, and we show that the flow of this vector field is linear on the invariant tori. Combining these results leads to the proof that the potentials $V_{\alpha\beta}$ define an a.c.i. system for $\alpha\neq\beta$ and we derive a Lax representation for it. Our proof of algebraic complete integrability is unusual in the sense that we do not use the Laurent solutions to the differential equations (see [AvM3]), nor the Lax equations (which often only come up at the end) (see [Gr]).

Do the Abelian surfaces generated by the potentials (1) account for all moduli of $(1,4)$-polarized Abelian surfaces? The answer is yes. In order to state precisely this answer (as given in Section 4), we first make a detailed study of the moduli space $\mathcal{A}_{(1,4)}$ of Abelian surfaces of type (1,4) and of some associated moduli spaces (Section 4). We use some results from [BLS] to construct a map ψ from $\mathcal{A}_{(1,4)}$ to an algebraic cone $\mathcal{M}^3$ of dimension 3, which lives in weighted projective space $\mathbb{P}^{(1,2,2,3,4)}$. The map is bijective on the dense subset $\tilde{\mathcal{A}}_{(1,4)}$ of Abelian surfaces for which the above map $\phi_{\mathcal{L}}$ is birational and the image is an affine variety $\mathcal{M}^3\setminus D$ where D is some divisor in $\mathcal{M}^3$; the two-dimensional subset $\mathcal{A}_{(1,4)}\setminus\tilde{\mathcal{A}}_{(1,4)}$ which consists of those Abelian surfaces $(\mathcal{T}^2,\mathcal{L})$ for which $\phi_{\mathcal{L}}$ is 2:1 however maps to a curve C (minus two points P,Q), which itself is a divisor in D. It follows that the image of the map $\psi\colon\mathcal{A}_{(1,4)}\to\mathbb{P}^{(1,2,2,3,4)}$ consists of the union

$$\mathcal{I}=(\mathcal{M}^3\setminus D)\,\cup\,(C\setminus\{P,Q\}),$$

and the cone $\mathcal{M}^3$ can be considered as a compactification of $\mathcal{A}_{(1,4)}$. Equations for $\mathcal{M}^3$, D, C and coordinates for the points P and Q will be explicitly calculated. We prove that for every point in the cone $\mathcal{M}^3$ (except for its vertex) there is at least one invariant surface of some potential $V_{\alpha\beta}$ corresponding to it under ψ (Theorem 3).

We also define a map from $\tilde{\mathcal{A}}_{(1,4)}$ onto the moduli space of two-dimensional Jacobians, or what is the same the moduli space of smooth curves of genus two. Namely we show (Section 5) that for every $\mathcal{T}^2 \in \tilde{\mathcal{A}}_{(1,4)}$ there exists exactly one Jacobi surface $J = J(\mathcal{T}^2)$ (with curve $\Gamma = \Gamma(\mathcal{T}^2)$) such that the map $\mathbf{2}_J$ (multiplication by 2 in J) factorizes over $\mathcal{T}^2$ (hence also over its dual $\hat{\mathcal{T}}^2$,) i.e., there is a commutative diagram

$$\begin{array}{ccc} J & \xrightarrow{4:1} & \hat{\mathcal{T}}^2 \\ \Big\downarrow{\scriptstyle 4:1} & {\scriptstyle \mathbf{2}_J}\searrow & \Big\downarrow{\scriptstyle 4:1} \\ \mathcal{T}^2 & \xrightarrow{4:1} & J \end{array} \tag{3}$$

We call this Jacobian the *canonical Jacobian* (of $\mathcal{T}^2$); it will also appear naturally in Section 3 when linearizing the vector field defined by the potentials $V_{\alpha\beta}$. One sees from the diagram that $\mathcal{T}^2$ cannot be reconstructed from J (or Γ); indeed $\mathcal{T}^2$ induces a decomposition $\Lambda = \Lambda_1 \oplus \Lambda_2$ of any lattice Λ defining $J = \mathbb{C}^2/\Lambda$ (and a partition $\mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2$ of the set of Weierstraß points of Γ) and this extra datum suffices to reconstruct $\mathcal{T}^2$ from J (or Γ). This will be shown in Section 5.

The problem arises to calculate this map explicitely as well as the extra data. We know of no direct algebraic way to do this. Instead we solve this problem (in Section 6) by relying heavily on the particular coordinates provided by the potentials $V_{\alpha\beta}$. Some geometrical investigations then lead to the following result: the curve $\Gamma(\mathcal{T}^2)$ corresponding to $\mathcal{T}^2$ is given by

$$y^2 = x(x-1)\left(4\lambda_2^2x^3 - (\lambda_0^2 + 2\lambda_1^2 + 6\lambda_2^2 + 2\lambda_3^2)x^2 + (\lambda_0^2 - 2\lambda_1^2 + 2\lambda_2^2 + 6\lambda_3^2)x - 4\lambda_3^2\right),$$

when the coordinate x is chosen such that it sends the points of $\mathcal{W}_2$ to $0, 1$ and ∞; $\mathcal{W}_1$ contains the other 3 Weierstraß points on this curve. We obtain this result in two different ways: one way uses the cover $J \to \mathcal{T}^2$ and the other uses the cover $\mathcal{T}^2 \to J$. It would be nice to calculate this map in a direct way, i.e., without using the $V_{\alpha\beta}$.

In the final section (Section 7) we study the degenerate case $V_{\alpha\alpha}$ as a limit of the generic case $V_{\alpha\beta}$ ($\alpha \neq \beta$). Since the potentials $V_{\alpha\alpha}$ are central they are obviously integrated using polar coordinates; these coordinates will be obtained as a limit of the linearizing variables for the generic case ($V_{\alpha\beta}$, $\alpha \neq \beta$) as well as the Lax representation (with a spectral parameter). This shows that the systematic techniques developped in [V1] to obtain linearizing variables and Lax equations for generic two-dimensional a.c.i. systems can lead to these data for integrable systems whose invariant manifolds are not Abelian varieties. We prove that in this degenerate case the affine invariant manifolds are $\mathbb{C}^*$-bundles over an elliptic curve, which itself is the spectral curve going with the Lax pair. Also we show that the invariant manifolds of all central potentials $V_{\alpha\alpha}$ corresponds to the special point $P \in \mathcal{M}^3$ at the boundary of $\mathcal{I}$.

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2. Preliminaries

In this section we recall some results about Abelian surfaces of type (1,4) which will be used in this paper (see [BLS], [GH], [LB]), as well as the basic techniques to study two-dimensional (algebraic) completely integrable systems (see [V1]).

2.1. Abelian surfaces of type (1,4)

Let Λ be a rank 4 lattice in $\mathbb{C}^2$, and form the associated complex torus $\mathcal{T}^2 = \mathbb{C}^2/\Lambda$. By a theorem of Riemann, $\mathcal{T}^2$ is an Abelian surface (i.e., can be embedded in projective space) if and only if there exists a complex base $\{e_1, e_2\}$ for $\mathbb{C}^2$ and an integer base $\{\lambda_1, \dots, \lambda_4\}$ for Λ such that the latter base can be written in terms of the former as

$$\Lambda = \begin{pmatrix} \delta_1 & 0 & a & b \\ 0 & \delta_2 & b & c \end{pmatrix}$$

(i.e., $\lambda_1 = \delta_1 e_1, \dots$) where $\delta_1 \mid \delta_2 \in \mathbb{N}$ and $\Im \begin{pmatrix} a & b \\ b & c \end{pmatrix} > 0$. The integers δ_1 and δ_2 are not invariants for the Abelian surface $\mathcal{T}^2$ itself, but for $\mathcal{T}^2$ equipped with some additional data: if $\mathcal{L}$ is an *ample line bundle* on $\mathcal{T}^2$ (i.e., a line bundle for which the sections of some power of the line bundle embeds the surface in projective space) then a base $\lambda_1, \dots, \lambda_4$ for Λ can be chosen such that the first Chern class $c_1(\mathcal{L})$ is given in terms of coordinates $x_1, \dots, x_4$, dual to $\lambda_1, \dots, \lambda_4$, by

$$c_1(\mathcal{L}) = \delta_1 dx_1 \wedge dx_3 + \delta_2 dx_2 \wedge dx_4.$$

$c_1(\mathcal{L})$ is called the *polarization determined by* $\mathcal{L}$ and depends only on $\mathcal{L}$ up to algebraic equivalence; δ_1 and δ_2 are invariants of $c_1(\mathcal{L})$. The pair (δ_1, δ_2) is called the *type* of $\mathcal{L}$, (or the *type of the polarization* $c_1(\mathcal{L})$). Loosely speaking we often say that the Abelian surface $\mathcal{T}^2$ has type (δ_1, δ_2). $\mathcal{T}^2$ is said to be principal polarized if it has type $(1,1)$. A principal polarized Abelian surface is either isomorphic to a product of elliptic curves (each taken with its principal polarization), or to the Jacobian of a smooth curve of genus two, polarized by its theta divisor Θ.

For a generic Abelian surface the line bundle $\mathcal{L} = [\mathcal{D}]$ corresponding to any effective divisor $\mathcal{D}$ is ample and one has the following useful string of identities:

$$g(\mathcal{D}) - 1 = \dim H^0\left(\mathcal{T}^2, \mathcal{O}(\mathcal{L})\right) = \delta_1\delta_2, \tag{4}$$

where $g(\mathcal{D})$ is the *virtual genus* of $\mathcal{D}$, which can (for Abelian surfaces) be defined in terms of intersection of divisors by

$$g(\mathcal{D}) = \frac{\mathcal{D}\cdot\mathcal{D}}{2} + 1; \tag{5}$$

if $\mathcal{D}$ is non-singular, $g(\mathcal{D})$ is just the topological genus of $\mathcal{D}$. To $\mathcal{L}$ there is associated a rational map $\phi_{\mathcal{L}}: \mathcal{T}^2 \to \mathbb{P}^{\delta_1\delta_2 - 1}$ which is defined by means of the sections of the sheaf $\mathcal{O}(\mathcal{L})$, or equivalently by means of the elements of $L(\mathcal{D})$, where

$$L(\mathcal{D}) = \{f \mid f \text{ meromorphic on } \mathcal{T}^2 \text{ and } (f) + \mathcal{D} \geq 0\}.$$

In this paper we concentrate on Abelian surfaces of type $(1,4)$. These Abelian surfaces have a very rich geometry, which we describe now (see [BLS]). As in [BLS] we will without further mention always restrict ourselves to those Abelian surfaces of type $(1,4)$ which are not isomorphic to a product of elliptic curves as polarized Abelian surfaces. Let $\mathcal{L}$ be a line bundle of type $(1,4)$ on an Abelian surface $\mathcal{T}^2$. It follows from (4) that $\dim H^0(\mathcal{T}^2,\mathcal{O}(\mathcal{L}))=4$ and $\mathcal{L}$ induces a rational map $\phi_{\mathcal{L}}:\mathcal{T}^2\to\mathbb{P}^3$.

- In the generic case, the image of this map $\mathcal{O}=\phi_{\mathcal{L}}(\mathcal{T}^2)\subset\mathbb{P}^3$ is an octic and $\phi_{\mathcal{L}}$ is birational on its image. Let $K(\mathcal{L})$ be the kernel of the isogeny

$$\begin{aligned} I_{\mathcal{L}}:\mathcal{T}^2 &\to \hat{\mathcal{T}}^2 \\ a &\mapsto t_a\mathcal{L}\otimes\mathcal{L}^{-1} \end{aligned}$$

between $\mathcal{T}^2$ and its dual $\hat{\mathcal{T}}^2$ (defined as the set of all line bundles on $\mathcal{T}^2$ of degree 0; t_a is translation by $a\in\mathcal{T}^2$), then $K(\mathcal{L})$ is a group of translations, isomorphic to $\mathbb{Z}/4\mathbb{Z}\oplus\mathbb{Z}/4\mathbb{Z}$. Picking any such isomorphism, let σ and τ be generators of the subgroups corresponding to this decomposition. Then homogeneous coordinates $(y_0:y_1:y_2:y_3)$ for $\mathbb{P}^3$ can be picked, such that σ,τ and the (-1)-involution $\imath$ on $\mathcal{T}^2$ (defined as $\imath(z_1,z_2)=(-z_1,-z_2)$ for $(z_1,z_2)\in\mathbb{C}^2/\Lambda$) act as follows (see [M1]):

$$\begin{aligned} \sigma(y_0:y_1:y_2:y_3) &= (y_2:y_3:y_0:-y_1), \\ \tau(y_0:y_1:y_2:y_3) &= (y_1:y_0:iy_3:iy_2), \\ \imath(y_0:y_1:y_2:y_3) &= (y_0:y_1:y_2:-y_3), \end{aligned} \tag{6}$$

(strictly speaking it may be necessary to replace τ by 3τ; it is easily checked that these coordinates exist only for (σ,τ) and $(3\sigma,3\tau)$ or for $(\sigma,3\tau)$ and $(3\sigma,\tau)$). [BLS] show that the octic $\mathcal{O}$ is given in these coordinates by

$$\begin{aligned} &\lambda_0^2y_0^2y_1^2y_2^2y_3^2+\lambda_1^2(y_0^4y_1^4+y_2^4y_3^4)+\lambda_2^2(y_0^4y_2^4+y_1^4y_3^4)+\lambda_3^2(y_0^4y_3^4+y_1^4y_2^4)+ \\ &2\lambda_1\lambda_2(y_0^2y_1^2+y_2^2y_3^2)(y_1^2y_3^2-y_0^2y_2^2)+2\lambda_1\lambda_3(y_0^2y_3^2-y_1^2y_2^2)(y_0^2y_1^2-y_2^2y_3^2)+ \\ &2\lambda_2\lambda_3(y_1^2y_2^2+y_0^2y_3^2)(y_1^2y_3^2+y_0^2y_2^2)=0, \end{aligned} \tag{7}$$

for some $(\lambda_0:\lambda_1:\lambda_2:\lambda_3)\in\mathbb{P}^3\setminus S$ where S is some divisor of $\mathbb{P}^3$ which we will determine later (Section 6.4). Remark that for any $\epsilon_i=\pm1$, the coordinates $(\epsilon_0y_0:\epsilon_1y_1:\epsilon_2y_2:\epsilon_0\epsilon_1\epsilon_2y_3)$ will also satisfy (6) and these are the only coordinates with this property. It is also seen that, if (σ,τ) is replaced by $(3\sigma,3\tau)$, then the coordinates $(y_0:y_1:y_2:y_3)$ are replaced by $(y_0:y_1:y_2:-y_3)$. Since the equation of $\mathcal{O}$ depends only on y_i^2 these choices do not affect the equation (7), so there is associated to a decomposition $K(\mathcal{L})=K_1\oplus K_2$ (where K_1 and K_2 are cyclic of order 4) an equation for $\mathcal{O}$. [BLS] also show that the polarized Abelian surface as well as the decomposition of $\mathcal{K}(\mathcal{L})$ can be recovered from (7) and that every octic of the type (7) (with $(\lambda_0:\lambda_1:\lambda_2:\lambda_3)\notin S$) is the image $\phi_{\mathcal{L}}(\mathcal{T}^2)$ of some $(1,4)$-polarized Abelian surface $(\mathcal{T}^2,\mathcal{L})$.

If we denote by $\tilde{\mathcal{A}}^0_{(1,4)}$ the moduli space of (isomorphism classes of) $(1,4)$-polarized Abelian surfaces for which $\phi_{\mathcal{L}}$ is birational, equipped with a decomposition of $K(\mathcal{L})$ as above, then it follows that

$$\tilde{\mathcal{A}}^0_{(1,4)}\cong\frac{\mathbb{P}^3\setminus S}{\lambda_0\sim-\lambda_0}. \tag{8}$$

Moreover, if we denote by K the subgroup of $K(\mathcal{L})$ of two-torsion elements,

$$K=\{0,2\sigma,2\tau,2\tau+2\sigma\},$$

then $\mathcal{T}^2/K$ is a principal polarized Abelian surface, which is the Jacobian of a curve of genus two; we call $\mathcal{T}^2/K$ the *canonical Jacobian* associated to $\mathcal{T}^2$. Recall that for a two-dimensional Jacobian J its Kummer surface is the image of $\phi_{[2\Theta]} \subset \mathbb{P}^3$, where Θ is the theta divisor of J. Then it is seen from (6) that an equation for the Kummer surface of $\mathcal{T}^2/K$ is given by the quartic Q in $\mathbb{P}^3$, obtained by replacing y_i^2 by z_i in the equation (7) for $\mathcal{O}$ and there is an obvious projection $\bar{p}\colon \mathcal{O} \to Q$. In fact, choosing the origin of $\mathcal{T}^2$ such that $\mathcal{L}$ becomes symmetric, $\mathcal{L}$ is the pull-back of a line bundle $\mathcal{N}$ on $\mathcal{T}^2/K$ of type (1,1) via the canonical projection

$$p\colon \mathcal{T}^2 \to \mathcal{T}^2/K,$$

and $\phi_{\mathcal{N}^2}$ induces the Kummer mapping; [BLS] prove that the following diagram commutes

$$\begin{array}{ccc} \mathcal{T}^2 & \xrightarrow{\phi_{\mathcal{L}}} & \mathcal{O} \\ \Big\downarrow p & & \Big\downarrow \bar{p} \\ \mathcal{T}^2/K & \xrightarrow{\phi_{\mathcal{N}^2}} & Q \end{array} \tag{9}$$

- If $\phi_{\mathcal{L}}$ is not birational, then it is 2:1 and $\phi_{\mathcal{L}}(\mathcal{T}^2)$ is a quartic in $\mathbb{P}^3$, given by one of the equations

$$\begin{aligned} \lambda_1(y_0^2y_1^2 + y_2^2y_3^2) + \lambda_2(y_1^2y_3^2 - y_0^2y_2^2) &= 0, \\ \lambda_1(y_2^2y_3^2 - y_0^2y_1^2) + \lambda_3(y_1^2y_2^2 - y_0^2y_3^2) &= 0, \\ \lambda_2(y_1^2y_3^2 + y_0^2y_2^2) + \lambda_3(y_1^2y_2^2 + y_0^2y_3^2) &= 0, \end{aligned}$$

depending on the choice of the decomposition; in this case the Abelian surface as well as the decomposition of $K(\mathcal{L})$ can only partly be recovered from these equations and $\mathcal{T}^2/K$ is a product of elliptic curves (in particular $\mathcal{T}^2$ is isogeneous to a product of elliptic curves). Squaring each of these equations we find equation (7) respectively with

$$\begin{aligned} &\begin{cases} \lambda_0^2 = 2(\lambda_2^2 + \lambda_3^2) \\ \lambda_1 = 0 \end{cases} && \lambda_2\lambda_3 \neq 0,\ \lambda_2^2 - \lambda_3^2 \neq 0, \\ &\begin{cases} \lambda_0^2 = -2(\lambda_1^2 + \lambda_3^2) \\ \lambda_2 = 0 \end{cases} && \lambda_1\lambda_3 \neq 0,\ \lambda_1^2 - \lambda_3^2 \neq 0, \\ &\begin{cases} \lambda_0^2 = 2(\lambda_1^2 - \lambda_2^2) \\ \lambda_3 = 0 \end{cases} && \lambda_1\lambda_2 \neq 0,\ \lambda_1^2 + \lambda_2^2 \neq 0, \end{aligned} \tag{10}$$

Summarizing, in the first case (the generic case), $\phi_{\mathcal{L}}(\mathcal{T}^2)$ is an octic, $\mathcal{T}^2/K$ is a Jacobian and $\mathcal{T}^2$ as well as the decomposition of $K(\mathcal{L})$ can be reconstructed from the octic; in the other case $\phi_{\mathcal{L}}(\mathcal{T}^2)$ is a quartic, $\mathcal{T}^2/K$ is a product of elliptic curves and $\mathcal{T}^2$ cannot be reconstructed from the quartic. The rational map $\phi_{\mathcal{L}}$ provides us with a natural surjective map

$$\psi^0\colon \mathcal{A}^0_{(1,4)} \to \Big((\mathbb{P}^3 \setminus S) \bigcup \text{(three rational curves in } S\text{, each missing eight points)} \Big) / (\lambda_0 \sim -\lambda_0),$$

where $\mathcal{A}^0_{(1,4)}$ denotes the moduli space of (isomorphism classes of) $(1,4)$-polarized Abelian surfaces together with a decomposition of $K(\mathcal{L})$ (as above). The map ψ^0 extends the bijection (8) defined on the dense subset $\tilde{\mathcal{A}}^0_{(1,4)}$ of $\mathcal{A}^0_{(1,4)}$ and maps the (two-dimensional) complement of $\tilde{\mathcal{A}}^0_{(1,4)}$ to the three rational curves, which are thought of as lying inside the boundary of $\psi^0(\tilde{\mathcal{A}}^0_{(1,4)})$, i.e., in S; the generic point of S however does not correspond to Abelian surfaces, but to surfaces which can be interpreted as degenerations of Abelian surfaces (see [BLS]).

2.2. Two-dimensional a.c.i. systems

We now recall the basic tools to study two-dimensional a.c.i. systems (see [AvM1], [V1]). At first, an *integrable system* on $(\mathbb{R}^{2n}, \omega)$ (ω may be any symplectic structure on $\mathbb{R}^{2n}$ but the case that ω is the standard symplectic structure will suffice for this paper) consists of a Hamiltonian vector field X_H, defined as

$$\omega(X_H, \cdot) = dH(\cdot),$$

for which there exist $n-1$ additional invariants, i.e., there are n independent, Poisson-commuting functions $H_1, \dots, H_n$ on $\mathbb{R}^{2n}$; *Poisson-commuting functions* $F, G \in C^\infty(\mathbb{R}^{2n})$, are by definition functions for which their *Poisson bracket* $\{F, G\}_\omega = \omega(X_F, X_G)$ vanishes. The intersection

$$\bigcap_{i=1}^{n} \{x \in \mathbb{R}^{2n} \mid H_i(x) = c_i\}$$

is by Poisson-commutativity invariant for the flows of all X_{H_i} and is smooth for generic values of $c = (c_1, \dots, c_n)$. By the well-known Arnold-Liouville Theorem, the compact connected components of these invariant manifolds are diffeomorphic to real tori (the non-compact components being diffeomorphic to cilindres, assuming that the flow of the vector fields X_{H_i} is complete on them); moreover the flows of the vector fields X_{H_i} are linear, when seen as flows on the tori (cilindres) using the diffeomorphism. n is called the *dimension* of the system.

A notable case — which appears most often in both the classical and recent, mathematical and physics literature — is the case that there exist coordinates $q_1, \dots, q_{2n}$ for $\mathbb{R}^{2n}$, in which all $H_i, (i = 1, \dots, n)$ as well as all brackets $\{q_i, q_j\}_\omega$, $(i, j = 1 \dots, 2n)$ are polynomials (strictly speaking, for the larger class of these examples $(\mathbb{R}^{2n}, \{\cdot,\cdot\}_\omega)$ is replaced by the more general Poisson manifold $(\mathbb{R}^m, \{\cdot,\cdot\})$, where $\{\cdot,\cdot\}$ does not necessarily come from a symplectic structure). Then the symplectic structure and the vector field are easily complexified, giving a Poisson commuting family of functions on $\mathbb{C}^{2n}$ and for generic $c = (c_1, \dots, c_n)$ (where the c_i may now also take values in $\mathbb{C}$) the invariant manifolds

$$\mathcal{A}_c = \bigcap_{i=1}^{n} \{x \in \mathbb{C}^{2n} \mid H_i(x) = c_i\}$$

are affine (algebraic) varieties. In such a situation, the integrable system will be called *algebraic completely integrable* if these generic invariant manifolds $\mathcal{A}_c$ are affine parts of an Abelian variety T^n_c, $\mathcal{A}_c = T^n_c \setminus \mathcal{D}_c$, where $\mathcal{D}_c$ is the minimal divisor where the coordinate functions (restricted to the invariant manifolds) blow up, and if the (complex) flow of the vector fields on T_c is linear (see [AvM3]).

In the two-dimensional case ($n = 2$) the invariant manifolds complete into Abelian surfaces by adding one or several (possibly singular) curves to the affine surfaces $\mathcal{A}_c$. In this case, the following algorithm, proposed in [V1] leads to an explicit linearization (i.e., integration) of the vector field X_H (steps (1) and (2) are due to Adler and van Moerbeke, see [AvM1]).

(1) Compute the first few terms of the Laurent solutions to the differential equations, and use these to construct an embedding of the generic invariant manifolds in projective space (see [AvM3], [V1] and [V2]).

(2) Deduce from the embedding the structure of the divisors $\mathcal{D}_c$ to be adjoined to the (generic) affine invariant manifolds $\mathcal{A}_c$ in order to complete them into Abelian surfaces. At this point the type of polarization induced by each irreducible component of $\mathcal{D}_c$ can also be determined.

(3) a) If one of the components of $\mathcal{D}_c$ is a smooth curve Γ_c of genus two, compute the image of the rational map

$$\phi_{[2\Gamma_c]}: T_c^2 \to \mathbb{P}^3$$

which is a singular surface in $\mathbb{P}^3$, the Kummer surface $\mathcal{K}_c$ of $\mathrm{Jac}(\Gamma_c)$.

b) Otherwise, if one of the components of $\mathcal{D}_c$ is a $d{:}1$ unramified cover $\mathcal{C}_c$ of a smooth curve Γ_c of genus two, $p: \mathcal{C}_c \to \Gamma_c$, the map p extends to a map $\bar{p}: T_c^2 \to \mathrm{Jac}(\Gamma_c)$. In this case, let $\mathcal{E}_c$ denote the (non-complete) linear system $\bar{p}^*|2\Gamma_c| \subset |2\mathcal{C}_c|$ which corresponds to the complete linear system $|2\Gamma_c|$ and compute now the Kummer surface $\mathcal{K}_c$ of $\mathrm{Jac}(\Gamma_c)$ as the image of

$$\phi_{\mathcal{E}_c}: T_c^2 \to \mathbb{P}^3.$$

c) Otherwise, change the divisor at infinity so as to arrive in case a) or b). This can always be done for a generic Abelian surface (i.e., for an Abelian surface which has no automorphisms except identity and the obvious (-1)-involution).

(4) Choose a Weierstraß point W on the curve Γ_c and coordinates $(z_0{:}z_1{:}z_2{:}z_3)$ for $\mathbb{P}^3$ such that $\phi_{[2\Gamma_c]}(W) = (0{:}0{:}0{:}1)$ in case (3) a) and $\phi_{\mathcal{E}_c}(W) = (0{:}0{:}0{:}1)$ in case (3) b). Then this point will be a singular point (node) for $\mathcal{K}_c$ and $\mathcal{K}_c$ has an equation

$$p_2(z_0, z_1, z_2)z_3^2 + p_3(z_0, z_1, z_2)z_3 + p_4(z_0, z_1, z_2) = 0,$$

where the p_i are polynomials of degree i. After a projective transformation which fixes $(0{:}0{:}0{:}1)$ we may assume that

$$p_2(z_0, z_1, z_2) = z_1^2 - 4z_0z_2.$$

(5) Finally, let x_1 and x_2 be the roots of the quadratic equation $z_0x^2 + z_1x + z_2 = 0$, whose discriminant is $p_2(z_0, z_1, z_2)$, with the z_i expressed in terms of the original variables $q_1, \ldots, q_4$. Then the differential equations describing the vector field X_H are rewritten by direct computation in the classical Weierstraß form

$$\frac{dx_1}{\sqrt{f(x_1)}} + \frac{dx_2}{\sqrt{f(x_2)}} = \alpha_1 dt,$$
$$\frac{x_1dx_1}{\sqrt{f(x_1)}} + \frac{x_2dx_2}{\sqrt{f(x_2)}} = \alpha_2 dt,$$

where α_1 and α_2 depend on c (i.e., on the torus) only. From it, the symmetric functions $x_1 + x_2\ (= -z_1/z_0)$ and $x_1x_2\ (= z_2/z_0)$ and hence also the original variables $q_1, \ldots, q_4$ can be written in terms of the Riemann theta function associated to the curve $y^2 = f(x)$.

The best way to see that this algorithm is very effective and easy to apply is to look at one or several of the worked-out examples in [V1]. In the present paper this algorithm will not be used as it stands, since we do not know in advance that our system is a.c.i.; instead we will see how it can be helpful when proving algebraic complete integrability. We remark that it is shown in [V1] how a Lax pair for the system derives from the above linearization.

3. The quartic potential $V_{\alpha\beta}$ and its integrability

It is shown in [CC] that for any $\lambda=(\lambda_1,\dots,\lambda_n)$, the potential

$$V_\lambda=\left(\sum_{i=1}^n q_i^2\right)^2+\sum_{i=1}^n \lambda_i q_i^2, \tag{11}$$

defines an integrable system on $\mathbb{R}^{2n}=\{(q_1,\dots,q_n,p_1,\dots,p_n)\mid q_i,p_i\in\mathbb{R}\}$, equipped with the standard symplectic structure $\omega=\sum dq_i\wedge dp_i$, when the Hamiltonian is taken as the total energy

$$H_\lambda=T+V_\lambda,\ T=\frac{1}{2}\sum_{i=1}^n p_i^2,$$

(T is the *kinetic energy*). This result also follows immediately from the integrability of the Garnier system, which will be recalled in the Appendix. We study here the case $n=2$ (two degrees of freedom) writing

$$V_{\alpha\beta}=(q_1^2+q_2^2)^2+\alpha q_1^2+\beta q_2^2.$$

It would be interesting to study also the higher-dimensional potentials as well as other cases of the Garnier system from the point of view of algebraic geometry.

Fixing arbitrary parameters $\alpha\neq\beta$, let $H=T+V_{\alpha\beta}$. Then the equations for the vector field X_H, defined by $\omega(X_H,\cdot)=dH(\cdot)$ are given by

$$\begin{aligned}
\dot q_1&=p_1, &\qquad \dot p_1&=-2q_1(2q_1^2+2q_2^2+\alpha),\\
\dot q_2&=p_2, &\qquad \dot p_2&=-2q_2(2q_1^2+2q_2^2+\beta).
\end{aligned} \tag{12}$$

For any f,g consider the affine surface $\mathcal{A}_{fg}$ defined by

$$\begin{aligned}
F&\equiv(q_1p_2-q_2p_1)^2+(\beta-\alpha)(p_1^2+2q_1^4+2q_1^2q_2^2+2\alpha q_1^2)=f,\\
G&\equiv(q_1p_2-q_2p_1)^2+(\alpha-\beta)(p_2^2+2q_2^4+2q_1^2q_2^2+2\beta q_2^2)=g,
\end{aligned}$$

(when the dependence on α and β is important we will denote this surface by $\mathcal{A}_c$ where $c=(\alpha,\beta,f,g)$). Then $\mathcal{A}_{fg}$ is invariant under the flow of X_H since both F and G Poisson commute with H. Since

$$F-G=2(\beta-\alpha)H$$

and $\alpha\neq\beta$, any pair of functions taken form $\{F,G,H\}$ can be taken as a maximal set of independent Poisson commuting functions; in order t o simplify some of the formulas in the sequel we let, for given f and g, the constant h be determined by $f-g=2(\beta-\alpha)h$.

The surface $\mathcal{A}_{fg}$ has the following independent involutions:

$$\begin{aligned}
\imath_1(q_1,q_2,p_1,p_2)&=(-q_1,q_2,-p_1,p_2),\\
\imath_2(q_1,q_2,p_1,p_2)&=(q_1,-q_2,p_1,-p_2),
\end{aligned}$$

which both preserve the vector field, and one other (independent) involution

$$\jmath(q_1,q_2,p_1,p_2)=(q_1,q_2,-p_1,-p_2),$$

which reverses the direction of the vector field. These three involutions generate a group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$. Moreover one sees that for fixed α, β, f and g all $\mathcal{A}_{(\lambda\alpha,\lambda\beta,\lambda^3 f,\lambda^3 g)}$, $\lambda \in \mathbb{C}^*$ are isomorphic. It is therefore natural to consider (α, β, f, g) as belonging to the weighted projective space[1] $\mathbb{P}^{(1,1,3,3)}$. A trivial observation which will turn out to be important is that also $\mathcal{A}_{(\alpha,\beta,f,g)}$ and $\mathcal{A}_{(\beta,\alpha,g,f)}$ are isomorphic.

Remark that if $\alpha = \beta$ then $F(=G)$ is just the square of the momentum

$$q = q_1 p_2 - q_2 p_1, \tag{13}$$

which obviously Poisson-commutes with the energy corresponding to a central potential. What is remarkable however is that if $\alpha \neq \beta$ then the equations defining $\mathcal{A}_{fg}$ can be rewritten (birationally) in terms of q_1, q_2 and the momentum q, giving precisely the equations (7) of the octic $\mathcal{O}$ with

$$\begin{aligned} \lambda_0^2 &= 4(\alpha-\beta)^2(\alpha+\beta) - 2(f+g), & y_0 &= 1, \\ \lambda_1^2 &= g, & y_1 &= q_1 \sqrt[4]{2(\alpha-\beta)/f}, \\ \lambda_2^2 &= 2(\alpha-\beta)^3, & y_2 &= q/\sqrt[4]{fg}, \\ \lambda_3^2 &= f, & y_3 &= q_2 \sqrt[4]{2(\alpha-\beta)/g}. \end{aligned} \tag{14}$$

It follows that for generic f, g the surface $\mathcal{A}_{fg}$ is birationally equivalent to the affine part $\mathcal{O}_0 = \mathcal{O} \cap \{y_0 \neq 0\}$ of the octic $\mathcal{O}$ which is itself birationally equivalent to an Abelian surface of type $(1,4)$. We show in the following theorem that $\mathcal{A}_{fg}$ actually *is* (isomorphic to) an affine part of an Abelian surface of type $(1,4)$.

Theorem 1 *Fixing any $\alpha \neq \beta \in \mathbb{C}$, the affine surface $\mathcal{A}_{fg} \subset \mathbb{C}^4$ defined by*

$$\begin{aligned} (q_1p_2 - q_2p_1)^2 + (\beta-\alpha)(p_1^2 + 2q_1^4 + 2q_1^2q_2^2 + 2\alpha q_1^2) &= f, \\ (q_1p_2 - q_2p_1)^2 + (\alpha-\beta)(p_2^2 + 2q_2^4 + 2q_1^2q_2^2 + 2\beta q_2^2) &= g, \end{aligned}$$

is for generic[2] $f, g \in \mathbb{C}$ isomorphic to an affine part of an Abelian surface $\mathcal{T}_{fg}^2$, of type $(1,4)$, obtained by removing a smooth curve $\mathcal{D}_{fg}$ of genus 5,

$$\mathcal{A}_{fg} = \mathcal{T}_{fg}^2 \setminus \mathcal{D}_{fg},$$

and the vector field X_H extends to a linear vector field on $\mathcal{T}_{fg}^2$.

Proof
(i) Let G be the group generated by the involutions $\imath_1, \imath_2$, and $\jmath$. Our first aim is to show that $\mathcal{A}_{fg}/G$ is (isomorphic to) an affine part of a Kummer surface. Since f and g are generic, we may suppose that $(\lambda_0\colon\lambda_1\colon\lambda_2\colon\lambda_3)$ given by (14) do not belong to S. For these λ_i, let Q be the quadric (Kummer surface)

$$\begin{aligned} &\lambda_0^2 z_0z_1z_2z_3 + \lambda_1^2(z_0^2z_1^2 + z_2^2z_3^2) + \lambda_2^2(z_0^2z_2^2 + z_1^2z_3^2) + \lambda_3^2(z_0^2z_3^2 + z_1^2z_2^2) + \\ &2\lambda_1\lambda_2(z_0z_1 + z_2z_3)(z_1z_3 - z_0z_2) + 2\lambda_1\lambda_3(z_0z_3 - z_1z_2)(z_0z_1 - z_2z_3) + \\ &2\lambda_2\lambda_3(z_1z_2 + z_0z_3)(z_1z_3 + z_0z_2) = 0, \end{aligned} \tag{15}$$

[1] a quick introduction to weighted projective spaces is given in an appendix to [AvM3]
[2] precise conditions will be given later (Theorem 6)

which is obtained from (7) by setting $z_i = y_i^2$, i.e., there is an unramified 8:1 cover $\mathcal{O} \to Q$; this map restricts to a map $\bar{p}_0: \mathcal{O}_0 \to Q_0$, where $Q_0 = Q \cap \{z_0 \neq 0\}$. Also the rational map $\phi: \mathcal{A}_{fg} \to \mathcal{O}_0$ given by (13) and (14) induces a birational map $\tilde{\phi}: \mathcal{A}_{fg}/G \to Q_0$, giving rise to a commutative diagram

$$\begin{array}{ccc} \mathcal{A}_{fg} & \xrightarrow{\phi} & \mathcal{O}_0 \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \bar{p}_0} \\ \mathcal{A}_{fg}/G & \xrightarrow{\tilde{\phi}} & Q_0 \end{array} \tag{16}$$

Since Q_0 is normal, it suffices to show that $\tilde{\phi}$ is bijective. Obviously $\tilde{\phi}$ is surjective: if $(x_1, x_2, x_3) \in Q_0$, let (y_1, y_2, y_3) be such that $y_i^2 = x_i$ and let q_1, q_2, q be determined from (14). Then these satisfy the condition under which p_1, p_2 exist such that $(q_1, q_2, p_1, p_2) \in \mathcal{A}_{fg}$ and $q = q_1 p_2 - q_2 p_1$. Then $\tilde{\phi}(q_1, q_2, p_1, p_2) = (x_1, x_2, x_3)$. At the other hand, if $(\tilde{\phi} \circ \pi)(q_1, q_2, p_1, p_2) = (\tilde{\phi} \circ \pi)(q_1', q_2', p_1', p_2')$ then $q_1 = \epsilon_1 q_1', q_2 = \epsilon_2 q_2', q = \epsilon q'$, (where $q' = q_1' p_2' - q_2' p_1'$) for $\epsilon_1, \epsilon_2, \epsilon \in \{-1, 1\}$. Then one sees that

$$(q_1, q_2, p_1, p_2) = \imath_1^{\epsilon_1} \imath_2^{\epsilon_2} \imath^{\epsilon} (q_1', q_2', p_1', p_2'),$$

where $i_k^{\epsilon_k}$ means $\imath_k$ in case $\epsilon_k = -1$ and identity for $\epsilon_k = 1$. It follows that $\pi(q_1, q_2, p_1, p_2) = \pi(q_1', q_2', p_1', p_2')$, and $\tilde{\phi}$ is injective. This shows that $\tilde{\phi}$ is an isomorphism, hence $\mathcal{A}_{fg}/G$ is isomorphic to the (affine) Kummer surface defined by Q_0.

(ii) We proceed to show that $\mathcal{A}_{fg}$ is isomorphic to an affine part of an Abelian surface, more precisely to the normalization $\mathcal{A}$ of $\mathcal{O}_0$ (the octic is singular along the coordinate planes). This normalization can be obtained via the birational map $\phi_{\mathcal{L}}: \mathcal{T}^2 \to \mathcal{O}$. In particular, by restriction of (9) to an affine piece we get a commutative diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\phi_{\mathcal{L}}} & \mathcal{O}_0 \\ \downarrow{\scriptstyle p_0} & & \downarrow{\scriptstyle \bar{p}_0} \\ K_0 & \xrightarrow{\phi_{\mathcal{N}^2}} & Q_0 \end{array} \tag{17}$$

where $\phi_{\mathcal{N}^2}$ is an isomorphism. If we combine both diagrams (16) and (17) we get

$$\begin{array}{ccc} \mathcal{A}_{fg} & \xrightarrow{\varphi} & \mathcal{A} \\ \downarrow{\scriptstyle 8:1} & & \downarrow{\scriptstyle 8:1} \\ \mathcal{A}_{fg}/G & \xrightarrow{\tilde{\varphi}} & K_0 \end{array}$$

with φ the birational map $\phi_{\mathcal{L}}^{-1}\phi$ and $\tilde{\varphi}$ the isomorphism $\phi_{\mathcal{N}^2}^{-1}\tilde{\phi}$. Now the two covers $\mathcal{A}_{fg} \to \mathcal{A}_{fg}/G$ and $\mathcal{A} \to K_0$ are only ramified in discrete points; the same holds true if $\mathcal{A}$ and $\mathcal{A}_{fg}$ are replaced by their closures: the closure of $\mathcal{A}$ is just $\mathcal{T}^2$ and the closure of $\mathcal{A}_{fg}$ is obtained from the explicit embedding which will be given in 6.1. By Zariski's Main Theorem the normality of $\mathcal{T}^2$ implies that the lifting φ of $\tilde{\varphi}$ must also be an isomorphism and we get

$$\mathcal{A}_{fg} = \mathcal{T}_{fg}^2 \setminus \mathcal{D}_{fg}$$

for some divisor $\mathcal{D}_{fg}$ on a $(1,4)$-polarized Abelian surface $\mathcal{T}_{fg}^2$. It is seen that $\mathcal{D}_{fg}$ is a 4:1 unramified cover of a translate of the Riemann theta divisor of the canonical Jacobian, hence $\mathcal{D}_{fg}$ is smooth and has genus 5; an equation for $\mathcal{D}_{fg}$ will be given in Section 6.

(iii) Finally we show that X_H extends to a linear vector field on $\mathcal{T}^2_{fg}$. Letting $\theta_0 = 1$, $\theta_1 = q_1^2$ and $\theta_3 = q^2$, we have shown that an equation for the Kummer surface of the canonical Jacobian associated to $\mathcal{A}_{fg}$ is a quartic in these variables. From (14) and (7) the leading term in θ_3^2 is given by $((\alpha+\beta)\theta_0+\theta_1+\theta_2)^2 - 4(\alpha\beta\theta_0+\beta\theta_1+\alpha\theta_2)$, or, in terms of the original variables,

$$(q_1^2+q_2^2+\alpha+\beta)^2 - 4(\alpha\beta+\alpha q_2^2+\beta q_1^2). \tag{18}$$

We let x_1 and x_2 be the roots of the polynomial

$$x^2 + \left(q_1^2+q_2^2+\alpha+\beta\right)x + \alpha\beta+\alpha q_2^2+\beta q_1^2,$$

as suggested by the algorithm recalled in Section 2.2 ("suggested" because we did not prove yet that the system is a.c.i.). Explicitely, let

$$\begin{aligned} x_1+x_2 &= -(q_1^2+q_2^2+\alpha+\beta), & \dot{x}_1+\dot{x}_2 &= -2(q_1p_1+q_2p_2),\\ x_1x_2 &= \alpha\beta+\alpha q_2^2+\beta q_1^2, & x_1\dot{x}_2+\dot{x}_1x_2 &= 2(\beta q_1p_1+\alpha q_2p_2), \end{aligned} \tag{19}$$

then it is not hard to rewrite the equations $F = f$, $G = g$, defining $\mathcal{A}_{fg}$, in terms of $x_1, x_2, \dot{x}_1, \dot{x}_2$. This gives

$$\dot{x}_i^2 = \frac{8(x_i+\alpha)(x_i+\beta)\left(x_i^3+(\alpha+\beta)x_i^2+(\alpha\beta-h)x_i+(\beta f-\alpha g)/2(\alpha-\beta)\right)}{(x_1-x_2)^2}$$

so that

$$\begin{aligned} \frac{dx_1}{\sqrt{f(x_1)}} + \frac{dx_2}{\sqrt{f(x_2)}} &= 0,\\ \frac{x_1dx_1}{\sqrt{f(x_1)}} + \frac{x_2dx_2}{\sqrt{f(x_2)}} &= 2\sqrt{2}dt, \end{aligned} \tag{20}$$

where

$$f(x) = (x+\alpha)(x+\beta)\left(x^3+(\alpha+\beta)x^2+(\alpha\beta-h)x+\frac{\beta f-\alpha g}{2(\alpha-\beta)}\right).$$

Integrating (20) we see that X_H is a linear vector field on $\mathcal{A}_{fg}$, which obviously extends to a linear vector field on $\mathcal{T}^2_{fg}$. From this expression the symmetric functions x_1+x_2 and x_1x_2, hence the variables q_1, q_2, p_1, p_2 can be written at once in terms of theta functions (see [M2]).

Remark that as a by-product we find an equation

$$y^2 = (x+\alpha)(x+\beta)\left(x^3+(\alpha+\beta)x^2+(\alpha\beta-h)x+\frac{\beta f-\alpha g}{2(\alpha-\beta)}\right). \tag{21}$$

for the curve whose Jacobian is the canonical Jacobian associated to $\mathcal{T}^2_{fg}$. ∎

The theorem leads to the following important corollary:

Corollary 2 *If $\alpha \neq \beta$ then the potential*

$$V_{\alpha\beta} = \left(q_1^2+q_2^2\right)^2 + \alpha q_1^2 + \beta q_2^2$$

defines an a.c.i. system (in the sense of [AvM1]*) on* $\mathbb{R}^4$ *with the canonical symplectic structure. A Lax representation of the vector field* X_H*, where* $H = \frac{1}{2}(p_1^2 + p_2^2) + V_{\alpha\beta}$*, is given by*

$$\frac{d}{dt}\begin{pmatrix} v(x) & u(x) \\ w(x) & -v(x) \end{pmatrix} = \sqrt{2}\left[\begin{pmatrix} v(x) & u(x) \\ w(x) & -v(x) \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ x - 2(q_1^2+q_2^2) & 0 \end{pmatrix}\right],$$

where

$$\begin{aligned} u(x) &= x^2 + (q_1^2 + q_2^2 + \alpha + \beta)x + \alpha\beta + \alpha q_2^2 + \beta q_1^2, \\ v(x) &= \frac{1}{\sqrt{2}}\left[(q_1 p_1 + q_2 p_2)x + (\beta q_1 p_1 + \alpha q_2 p_2)\right], \\ w(x) &= x^3 + (\alpha + \beta - q_1^2 - q_2^2)x^2 - \left(\frac{p_1^2+p_2^2}{2} + (\alpha+\beta)\left(q_1^2+q_2^2\right) - \alpha\beta\right)x \\ &\qquad - \alpha\beta\left(\frac{p_1^2}{2\alpha} + \frac{p_2^2}{2\beta} + q_1^2 + q_2^2\right). \end{aligned}$$

Proof

The Liouville integrability is proven in [G] and [CC]; it is in our case proven easily by showing that $\{F, G\} = 0$ (F, G Poisson commute) and that F and G are independent on a dense subset of $\mathbb{R}^4$. To show that for $\alpha \neq \beta$ the system is a.c.i. we need to prove in addition the following three claims:

(i) the generic (complex) affine invariant surface $\mathcal{A}_{fg}$ is an affine part of an Abelian surface $\mathcal{T}^2_{fg}$, $\mathcal{A}_{fg} = \mathcal{T}^2_{fg} \setminus \mathcal{D}_{fg}$, where $\mathcal{D}_{fg}$ is some divisor on $\mathcal{T}^2_{fg}$,

(ii) $\mathcal{D}_{fg}$ is the minimal divisor where the variables q_1, q_2, p_1 and p_2 blow up,

(iii) the vector fields X_F and X_H extend to holomorphic (= linear) vector fields on $\mathcal{T}^2_{fg}$.

(i) and half of (iii) are shown in Theorem 1. To show the other half of (iii), which concerns the extension of X_F, the linearizing variables are defined in the same way, but their derivatives are now calculated using X_F instead of X_H. Finally, since the variables q_1, q_2, p_1 and p_2 do not blow up on $\mathcal{A}_{fg}$, and since $\mathcal{D}_{fg}$ is irreducible, they all blow up along $\mathcal{D}_{fg}$, showing (ii).

To construct a Lax pair, note that if $u(x)$ is defined as $u(x) = (x - x_1)(x - x_2)$ and $v(x)$ is its derivative (suitable normalised), then

$$f(x) - v^2(x) \text{ is divisible by } u(x),$$

where $f(x)$ is the polynomial introduced in the proof of Theorem 1. The quotient

$$w(x) = \frac{f(x) - v^2(x)}{u(x)}$$

is easily calculated. The form of the Lax pair then follows from [V1]. ∎

4. Some moduli spaces of Abelian surfaces of type (1,4)

In this section we describe a map ψ from the moduli space $\mathcal{A}_{(1,4)}$ of polarized Abelian surfaces of type $(1,4)$ into an algebraic cone $\mathcal{M}^3$ in some weighted projective space. To be precise we recall that $(1,4)$-polarized Abelian surfaces which are products of elliptic curves (with the product polarization) are excluded from $\mathcal{A}_{(1,4)}$. The map will be bijective on the dense subset $\tilde{\mathcal{A}}_{(1,4)}$ which is the moduli space of polarized Abelian surfaces $(\mathcal{T}^2,\mathcal{L})$ for which the rational map $\phi_{\mathcal{L}}:\mathcal{T}^2\to\mathbb{P}^3$ is birational. An alternative way to construct the map ψ and the cone $\mathcal{M}^3$ will come up later.

Recall from Section 2 that $\mathcal{A}^0_{(1,4)}$ maps onto

$$\mathcal{P}=\frac{\mathbb{P}^3\setminus S}{\lambda_0\sim-\lambda_0}\bigcup\text{(three rational curves in } S\text{, each missing eight points)},$$

bijectively on the first component (which is dense); the three rational curves are thought of as lying in $\mathbb{P}^3/(\lambda_0\sim-\lambda_0)$ at the boundary of this component. $\mathcal{A}^0_{(1,4)}$ is a $24{:}\,1$ (ramified) covering of $\mathcal{A}_{(1,4)}$: let σ and τ be elements of order 4 such that $K(\mathcal{L})=\langle\sigma\rangle\oplus\langle\tau\rangle$, and define

$$\begin{aligned}
K_1&=\{0,\sigma,2\sigma,3\sigma\}, & K_4&=\{0,\sigma+2\tau,2\sigma,3\sigma+2\tau\},\\
K_2&=\{0,\tau,2\tau,3\tau\}, & K_5&=\{0,2\sigma+\tau,2\tau,2\sigma+3\tau\},\\
K_3&=\{0,\sigma+\tau,2\sigma+2\tau,3\sigma+3\tau\}, & K_6&=\{0,\sigma+3\tau,2\sigma+2\tau,3\sigma+\tau\}.
\end{aligned}$$

These are the only cyclic subgroups of order 4 of $K(\mathcal{L})$. It is easy to see that taking all possible isomorphisms $K(\mathcal{L})\cong\mathbb{Z}/4\mathbb{Z}\oplus\mathbb{Z}/4\mathbb{Z}$ we find exactly the 24 decompositions

$$K(\mathcal{L})=K_i\oplus K_j,\ (1\le i,j\le 6,\ |i-j|\neq 0,3).$$

We describe the cover

$$\mathcal{A}^0_{(1,4)}\xrightarrow{24:1}\mathcal{A}_{(1,4)}$$

and construct a $24{:}\,1$ cover $\mathcal{P}\to\mathcal{M}^3$ and a map $\psi:\mathcal{A}_{(1,4)}\to\mathcal{M}^3$, where $\mathcal{M}^3$ is an algebraic variety (lying in weighted projective space $\mathbb{P}^{(1,2,2,3,4)}$), such that there results a commutative diagram

$$\begin{array}{ccccc}
\mathcal{A}^0_{(1,4)} & \xrightarrow{24:1} & \mathcal{A}_{(1,4)} & \supset & \tilde{\mathcal{A}}_{(1,4)}\\
\downarrow{\scriptstyle\psi^0} & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\tilde{\psi}}\\
\mathcal{P} & \xrightarrow{24:1} & \mathcal{M}^3 & \supset & \mathcal{M}^3\setminus D
\end{array}\tag{22}$$

in which the restriction $\tilde{\psi}$ of ψ to $\tilde{\mathcal{A}}_{(1,4)}$ is a bijection (D is a divisor on $\mathcal{M}^3$ which will be determined explicitely).

The main idea in this construction is to see how the Galois group of the cover $\mathcal{A}^0_{(1,4)}\to\mathcal{A}_{(1,4)}$ acts on $\mathcal{P}$ and define $\mathcal{M}^3$ to be the quotient. This quotient will be easy to calculate since it is a quotient of (a Zariski open subset of) $\mathbb{P}^3$ by a group which acts linearly. The fact that this action is so simple is surprizing and was suggested to us by the obvious observation that the affine invariant surfaces $\mathcal{A}_c$ and $\mathcal{A}_{c'}$, with $c=(\alpha,\beta,f,g)$ and $c'=(\beta,\alpha,g,f)$ are isomorphic, showing by (14) that λ_1 and λ_2 can (in some way) be interchanged.

The group $G = GL(2, \mathbb{Z}/4\mathbb{Z})$ acts transitively on (ordered!) bases as follows: if σ, τ are such that $K(\mathcal{L}) = \langle\sigma\rangle \oplus \langle\tau\rangle$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (\sigma, \tau) = (a\sigma + b\tau, c\sigma + d\tau),$$

giving a new decomposition $K(\mathcal{L}) = \langle a\sigma + b\tau\rangle \oplus \langle c\sigma + d\tau\rangle$. We denote by H the normal subgroup of G which consists of those elements of G which are congruent to the identity matrix, modulo 2. Then H acts on the set of decompositions of $K(\mathcal{L})$, thus H acts on $\mathcal{A}^0_{(1,4)}$; to determine the corresponding action on the isomorphic space $\mathcal{P}$, it is sufficient to take any element of H, act to obtain a new base and determine the new coordinates $(y_0{:}\, y_1{:}\, y_2{:}\, y_3)$ according to (6). Substituting these in (7) the new parameters $(\pm\lambda_0{:}\, \lambda_1{:}\, \lambda_2{:}\, \lambda_3)$ are found immediately. The result is contained in the following table (since diagonal matrices act trivially only one representative of each coset modulo diagonal matrices is shown):

H	base	$K(\mathcal{L})$	coo. for $\mathbb{P}^3$	moduli in $\mathcal{P}$
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	(σ, τ)	$K_1 \oplus K_2$	$(y_0{:}\, y_1{:}\, y_2{:}\, y_3)$	$(\pm\lambda_0{:}\, \lambda_1{:}\, \lambda_2{:}\, \lambda_3)$
$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$	$(\sigma + 2\tau, \tau)$	$K_4 \oplus K_2$	$(y_0{:}\, y_1{:}\, iy_2{:}\, iy_3)$	$(\pm\lambda_0{:}\, -\lambda_1{:}\, \lambda_2{:}\, \lambda_3)$
$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$	$(\sigma, 2\sigma + \tau)$	$K_1 \oplus K_5$	$(y_0{:}\, iy_1{:}\, y_2{:}\, iy_3)$	$(\pm\lambda_0{:}\, \lambda_1{:}\, -\lambda_2{:}\, \lambda_3)$
$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$	$(\sigma + 2\tau, 2\sigma + \tau)$	$K_4 \oplus K_5$	$(y_0{:}\, iy_1{:}\, iy_2{:}\, -y_3)$	$(\pm\lambda_0{:}\, \lambda_1{:}\, \lambda_2{:}\, -\lambda_3)$

Table 1

The upshot of the table is that all $(\pm\lambda_0{:}\, \pm\lambda_1{:}\, \pm\lambda_2{:}\, \pm\lambda_3)$ correspond to the same Abelian surface. The quotient space is given by

$$\begin{aligned} \mathcal{P}' &= \frac{\mathcal{P}}{(\pm\lambda_0{:}\, \lambda_1{:}\, \lambda_2{:}\, \lambda_3) \sim (\pm\lambda_0{:}\, \pm\lambda_1{:}\, \pm\lambda_2{:}\, \pm\lambda_3)} \\ &\cong \left(\mathbb{P}^3 \setminus S'\right) \bigcup \text{(three rational curves in } S'\text{, each missing three points)}, \end{aligned} \tag{23}$$

upon defining $\mu_i = \lambda_i^2$ as coordinates for the quotient $\mathbb{P}^3$, from which in particular equations for the three rational curves, as well as for the three points are immediately obtained (the fact that there are three missing points instead of two is due to ramification of the quotient map at two of the three points). The divisors S and S' will be calculated later. We will also interpret this "intermediate" moduli space $\mathcal{P}'$.

Remark that G/H is isomorphic to the permutation group S_3, so we have an action of S_3 on $\mathcal{P}'$ (which extends to all of $\mathbb{P}^3$ since it is linear). Choosing six representatives for G/H we find as above the following table:

S_3	G/H	base	$K(\mathcal{L})$	coo. for $\mathbb{P}^3$	moduli in $\mathcal{P}'$
$()$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	(σ, τ)	$K_1 \oplus K_2$	$(y_0 : y_1 : y_2 : y_3)$	$(\mu_0 : \mu_1 : \mu_2 : \mu_3)$
(12)	$\begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix}$	$(\tau, 3\sigma)$	$K_2 \oplus K_1$	$(y_0 : y_2 : y_1 : iy_3)$	$(-\mu_0 : \mu_2 : \mu_1 : \mu_3)$
(13)	$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	$(\sigma, \sigma+\tau)$	$K_1 \oplus K_3$	$(\sqrt{i}y_2 : y_1 : \sqrt{i}y_0 : y_3)$	$(\mu_0 : \mu_3 : -\mu_2 : \mu_1)$
(23)	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$(\sigma+\tau, \tau)$	$K_3 \oplus K_2$	$(y_1 : y_0 : \sqrt{i}y_2 : \sqrt{i}y_3)$	$(\mu_0 : -\mu_1 : \mu_3 : \mu_2)$
(123)	$\begin{pmatrix} 0 & 3 \\ 1 & 1 \end{pmatrix}$	$(3\tau, \sigma+\tau)$	$K_2 \oplus K_3$	$(\sqrt{i}y_1 : iy_2 : \sqrt{i}y_0 : y_3)$	$(\mu_0 : -\mu_3 : \mu_1 : -\mu_2)$
(321)	$\begin{pmatrix} 1 & 1 \\ 3 & 0 \end{pmatrix}$	$(\sigma+\tau, 3\sigma)$	$K_3 \oplus K_1$	$(\sqrt{i}y_2 : \sqrt{i}y_0 : -y_1 : -iy_3)$	$(\mu_0 : \mu_2 : -\mu_3 : -\mu_1)$

Table 2

The tables 1 and 2 together show how to reconstruct explicitly the decomposition of $K(\mathcal{L})$ from the equation of the octic. More important, it allows us to construct the quotient space $\mathcal{M}^3$ as is shown in the following theorem.

Theorem 3 *There is a bijective map $\tilde{\psi}: \tilde{\mathcal{A}}_{(1,4)} \to \mathcal{M}^3 \setminus D$, where $\mathcal{M}^3$ is the cone defined by*

$$f_4^2 = f_1(4f_2^3 - 27f_3^2)$$

in weighted projective space $\mathbb{P}^{(1,2,2,3,4)}$ (with coordinates $(f_0 : \cdots : f_4)$) and $D = D_1 + D_2$ is the divisor whose two irreducible components are cut off from $\mathcal{M}^3$ by the hypersurfaces

$$\begin{aligned} &D_1 : f_4 = f_1(f_1 - 3f_2), \\ &D_2 : 512 f_4 = -16\left(16f_2^2 + 72 f_1 f_2 - 27 f_1^2 - 48 f_0 f_3\right) + 3 f_0^2\left(f_0^2 + 24 f_1 - 32 f_2\right). \end{aligned} \tag{24}$$

In particular the moduli space $\tilde{\mathcal{A}}_{(1,4)}$ has the structure of an affine variety. The map $\tilde{\psi}$ extends in a natural way to a map

$$\psi: \mathcal{A}_{(1,4)} \to \mathcal{M}^3,$$

the image of the (two-dimensional) boundary $\mathcal{A}_{(1,4)} \setminus \tilde{\mathcal{A}}_{(1,4)}$ being $C \setminus \{P, Q\}$, where C is the rational curve (inside D) given by

$$C : 3f_0^2 = 4(4f_2 - f_1),$$

and $P, Q \in C$ are given by $P = (4:0:3:2:0)$, and $Q = (2:1:1:0:-2)$. Moreover, apart from its top $(1:0:0:0:0)$, all points in the cone $\mathcal{M}^3$ correspond to some invariant surface $\mathcal{A}_{(\alpha,\beta,f,g)}$ for some α, β, f and g, with $\alpha \neq \beta$.

Proof

First we describe the quotient of $\mathbb{P}^3$ by the action of S_3, and show that it is (isomorphic to) the algebraic variety $\mathcal{M}^3$ given by an equation $f_4^2 = f_1(4f_2^3 - 27f_3^2)$ in weighted projective space

$\mathbb{P}^{(1,2,2,3,4)}$. To do this we use the (induced) action of S_3 on $\mathbb{C}^3$ which is given in terms of affine coordinates $x_i = \mu_i/\mu_0$ for $\mathbb{C}^3$ by

$$\begin{aligned}(1,2)\cdot(x_1,x_2,x_3) &= (-x_2,-x_1,-x_3),\\ (1,2,3)\cdot(x_1,x_2,x_3) &= (-x_3,x_1,-x_2).\end{aligned}$$

Since the action is orthogonal, it must be reducible, having an invariant line and an invariant plane orthogonal to it. Indeed let

$$\begin{aligned}u_1 &= x_1 + x_2 - x_3,\\ u_2 &= x_1 - x_2,\\ u_3 &= x_1 + x_3,\end{aligned} \tag{25}$$

then u_1 is anti-invariant for $(1,2)$ and is invariant for $(1,2,3)$; u_2 and u_3 are chosen orthogonal to u_1. Then invariants

$$\begin{aligned}f_2 &= u_2^2 - u_2u_3 + u_3^2,\\ f_3 &= u_2u_3(u_2-u_3),\end{aligned}$$

for the action of S_3 are found. Also there is

$$\Delta = u_2^2(2u_2 - 3u_3) + u_3^2(2u_3 - 3u_2)$$

which is $(1,2)$-anti-invariant and $(1,2,3)$-invariant, giving a new invariant $f_4 = u_1\Delta$. Since f_2 and f_3 generate the invariants depending on u_2, u_3 the invariant Δ^2 is expressible in terms of f_2 and f_3,

$$\Delta^2 = 4f_2^3 - 27f_3^2,$$

i.e., Δ^2 is nothing else than the discriminant of the cubic polynomial $x^3 - f_2x + f_3$. It follows that

$$f_4^2 = f_1(4f_2^3 - 27f_3^2), \tag{26}$$

where $f_1 = u_1^2$. Remark that (f_1, f_2, f_3, f_4) have degree $(2,2,3,4)$ so that the quotient of $\mathbb{P}^3$ by the action of S_3 is given by (26) viewed as an equation in weighted projective space $\mathbb{P}^{(1,2,2,3,4)}$ with respect to coordinates $(f_0\colon f_1\colon f_2\colon f_3\colon f_4)$. In conclusion we have established the cover $\mathcal{P} \to \mathcal{M}^3$ and there is an induced map $\psi\colon \mathcal{A}_{(1,4)} \to \mathcal{M}^3$ which makes

$$\begin{array}{ccc} \mathcal{A}^0_{(1,4)} & \overset{24:1}{\longrightarrow} & \mathcal{A}_{(1,4)} \\ \big\downarrow{\scriptstyle \psi^0} & & \big\downarrow{\scriptstyle \psi} \\ \mathcal{P} & \overset{24:1}{\longrightarrow} & \mathcal{M}^3 \end{array} \tag{27}$$

into a commutative diagram (since the actions on $\mathcal{A}^0_{(1,4)}$ are the same by construction).

The reducible divisor D is easily computed once explicit equtions for S (or S') are known. Since we know of no easy direct way to determine S, we postpone the computation of S to Section 6.4, where the potentials will be used to compute S in a straightforward way; we will show there that S' breaks up in four irreducible pieces $\mu_1 = 0$, $\mu_2 = 0$, $\mu_3 = 0$ and $\mathrm{disc}(P_3^{\mu}(x)) = 0$ where P_3 is the polynomial

$$P_3 = 4\mu_2x^3 - (\mu_0 + 2\mu_1 + 6\mu_2 + 2\mu_3)x^2 + (\mu_0 - 2\mu_1 + 2\mu_2 - 6\mu_3)x - 4\mu_3,$$

and $\mathrm{disc}(P_3^\mu(x)) = 0$ denotes its discriminant (in x). Granted this, we take $\mu_1 = 0$, let $x_1 = 0$ and eliminate x_2 and x_3 from f_1, f_2 and f_4. Then the relation

$$f_4 = f_1(f_1 - 3f_2),$$

is found at once; obviously the same equation is found for $\mu_2 = 0$, $\mu_3 = 0$. The computation for $\mathrm{disc}(P_3^\mu(x)) = 0$ is longer but also straightforward. Namely, by a simple translation in x the monic polynomial $P_3^\mu(x)/(4\mu_2)$ can be written as $x^3 - ax + b$, with discriminant $4a^3 - 27b^2$. When this discriminant (depending on μ_i) is written in terms of u_i using the inverse of (25), the equation (24) for D_2 is read off immediately.

As for the curve to be added to $\tilde\psi(\tilde{\mathcal{A}}_{(1,4)})$ to obtain $\psi(\mathcal{A}_{(1,4)})$ remark that the action of S_3 identifies the three rational curves in (23), leading to a single curve. To compute its equation (as a subvariety of D_1) in terms of the coordinates f_i, let according to (10), $\mu_1 = 0$ and $\mu_0 = 2(\mu_2 + \mu_3)$. Then in terms of μ_0 and μ_2 we get

$$\begin{aligned} f_0 &= \mu_0, \\ f_1 &= (2\mu_2 - \mu_0/2)^2, \\ f_2 &= \mu_2^2 - \frac{\mu_0\mu_2}{2} + \frac{\mu_0^2}{4}, \end{aligned}$$

leading to

$$3f_0^2 = 4(4f_2 - f_1),$$

by elimination of μ_0 and μ_2. As for the two special points P and Q on this curve, it is easy to check that picking $\mu_1 = 0$, $\mu_2 = \mu_3$ and $\mu_0 = 2(\mu_2 + \mu_3)$ leads to the point $(4{:}\,0{:}\,3{:}\,2{:}\,0)$ and alternatively taking $\mu_1 = \mu_2 = 0$, $\mu_0 = 2\mu_3$ leads to the point $(2{:}\,1{:}\,1{:}\,0{:}\,-2)$. This gives explicit equations for all these spaces and proves the announced result in (22).

Finally, let $(f_0{:}\cdots{:}\,f_4) \in \mathcal{M}^3$ be any point different from the top $(1{:}\,0{:}\,0{:}\,0{:}\,0)$ of this cone. Then $\mu_2 \neq 0$ for at least one of the six points $(\mu_0{:}\,\mu_1{:}\,\mu_2{:}\,\mu_3)$ lying over this point. Define α, β, f, g by

$$\begin{aligned} \alpha &= \mu_0 + 2\mu_1 + 2\mu_2 + 2\mu_3, \\ \beta &= \mu_0 + 2\mu_1 - 2\mu_2 + 2\mu_3, \\ f &= 128\mu_2^2\mu_3, \\ g &= 128\mu_2^2\mu_1, \end{aligned} \tag{28}$$

then $\alpha \neq \beta$ and α, β, f and g satisfy (14). This shows that, apart from the top, all points in the cone $\mathcal{M}^3$ correspond to some invariant surface $\mathcal{A}_{(\alpha,\beta,f,g)}$ for some $\alpha \neq \beta$, f and g. This finishes the proof of the theorem. ∎

5. The precise relation with the canonical Jacobian

In this section we want to show that a (1,4)-polarized Abelian surface $\mathcal{T}^2 \in \tilde{\mathcal{A}}_{(1,4)}$ is intimely related to its canonical Jacobian, denoted by $J(\mathcal{T}^2)$ (introduced in Section 2), hence also to some curve of genus two, denoted $\Gamma(\mathcal{T}^2)$. In fact there is more: at the level of the Jacobian, let $J(\mathcal{T}^2)$ be represented as $\mathbb{C}^2/\Lambda$, then $\mathcal{T}^2$ induces a non-degenerate decomposition of the lattice Λ and at the level of the curve, $\mathcal{T}^2$ induces a decomposition of the set of Weierstraß points of $\Gamma(\mathcal{T}^2)$ which in turn corresponds to an incidence diagram for the 16_6 configuration on its Kummer surface; moreover, the Abelian surface can be reconstructed from either of these data (Theorem 4).

Recall that the canonical Jacobian of a (1,4)-polarized Abelian surface $\mathcal{T}^2 = (\mathcal{T}^2, \mathcal{L}) \in \tilde{\mathcal{A}}_{(1,4)}$ is defined as the (irreducible principally polarized) Abelian surface $J(\mathcal{T}^2) = \mathcal{T}^2/K$, where K is the (unique) subgroup of two-torsion elements of $K(\mathcal{L})$. As is well-known such an Abelian surface is the Jacobian of a smooth curve Γ of genus two, i.e., it is given as $\mathbb{C}^2/\Lambda$, where Λ is the *period lattice*

$$\Lambda = \left\{ \oint_\gamma \vec{\omega} \mid \gamma \in H_1(\Gamma, \mathbb{Z}) \right\}$$

consisting of all periods of $\vec{\omega} = {}^t(\omega_1, \omega_2)$, the ω_i being (independent) holomorphic differentials on Γ. The Abelian group $H_1(\Gamma, \mathbb{Z})$ has an (alternating) intersection form $\sharp(\cdot)$ and $H_1(\Gamma, \mathbb{Z})$ can be decomposed into non-degenerate planes (in many different ways),

$$H_1(\Gamma, \mathbb{Z}) = H_1 \oplus H_2, \qquad \sharp(\cdot)_{H_1} \text{ and } \sharp(\cdot)_{H_2} \text{ non-degenerate.}$$

Such a decomposition leads to a decomposition $\Lambda = \Lambda_1 \oplus \Lambda_2$ upon defining

$$\Lambda_i = \left\{ \oint_\gamma \vec{\omega} \mid \gamma \in H_i \right\}; \tag{29}$$

both $H_1(\Gamma, \mathbb{Z}) = H_1 \oplus H_2$ and $\Lambda = \Lambda_1 \oplus \Lambda_2$ will be called *non-degenerate decompositions.* They are called in addition *simple* if each H_i is generated by cycles which come from simple closed curves (Jordan curves) in $\mathbb{P}^1$ under some (hence any) double cover $\pi: \Gamma \to \mathbb{P}^1$.

We also recall from the classical literature the 16_6 configuration on the Kummer surface of $\mathrm{Jac}(\Gamma)$, where Γ is a curve of genus two. Let $W_1, \ldots, W_6$ be the Weierstraß points on Γ, then the points

$$W_{ij} = \int_{W_i}^{W_j} \vec{\omega} \pmod{\Lambda}$$

are half-periods of $\mathrm{Jac}(\Gamma)$, sixteen in total since $W_{ij} = W_{ji}$ and $W_{ii} = W_{jj}$ for all $i, j = 1, \ldots, 6$. There are also sixteen genus two curves Γ_{ij} in $\mathrm{Jac}(\Gamma)$, the translates $W_{ij} + \Gamma_{kk}$ of the single curve $\Gamma_{11} = \cdots = \Gamma_{66}$, which have the property that Γ_{11}, hence all Γ_{ij} pass through six points W_{kl}. Then also each point belongs to six lines Γ_{ij}. This whole configuration goes down to the Kummer surface in $\mathbb{P}^3$ and gives there a 16_6 configuration, classically called *Kummer's configuration.* The sixteen points are nodes (singular points) and the sixteen planes the lines belong to are tropes (singular planes) of the Kummer surface. The 16_6 configuration is best visualized by the *incidence diagram*, which consists of a pair of square diagrams, such as

$$\begin{array}{cccc} W_{11} & W_{12} & W_{23} & W_{13} \\ W_{45} & W_{36} & W_{16} & W_{26} \\ W_{46} & W_{35} & W_{15} & W_{25} \\ W_{56} & W_{34} & W_{14} & W_{24} \end{array} \qquad \begin{array}{cccc} \Gamma_{11} & \Gamma_{12} & \Gamma_{23} & \Gamma_{13} \\ \Gamma_{45} & \Gamma_{36} & \Gamma_{16} & \Gamma_{26} \\ \Gamma_{46} & \Gamma_{35} & \Gamma_{15} & \Gamma_{25} \\ \Gamma_{56} & \Gamma_{34} & \Gamma_{14} & \Gamma_{24} \end{array}$$

Namely the points incident with a line at position (m, n) in the second square diagram are those six points in the m-th row and n-th column, but not in both, of the first square diagram. Dually, the same applies for the lines incident with a point. The 24^2 incidence diagrams obtained by permuting the rows or columns of both square diagrams in an incidence diagram (in the same way) are defined to be the same as the original incidence diagram (we will see that there are 20 incidence diagrams which are different in this sense).

The relevance of simple, non-degenerate decompositions and incidence diagrams for (1,4)-polarized Abelian surfaces is seen from the following theorem.

Theorem 4 *There is a natural correspondence between the following (isomorphism classes) of data:*

(1) *a* $(1,4)$*-polarized Abelian surface* $\mathcal{T}^2 \in \tilde{\mathcal{A}}_{(1,4)}$,

(2) *a Jacobi surface* $J = \mathbb{C}^2/\Lambda$ + *a simple, non-degenerate decomposition* $\Lambda = \Lambda_1 \oplus \Lambda_2$ *of* Λ,

(3) *a smooth genus two curve* Γ + *a decomposition* $\mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2$, $\#\mathcal{W}_1 = \#\mathcal{W}_2 = 3$, *of its Weierstraß points.*

(4) *a smooth genus two curve* Γ + *an incidence diagram for the* 16_6 *configuration on its corresponding Kummer surface.*

The correspondence $(1) \leftrightarrow (2)$ *is established in two ways, namely* J *may be taken as the quotient of* $\mathcal{T}^2$ *using* Λ_2 *or as a cover of* $\mathcal{T}^2$ *using* Λ_1 *(or* $\mathcal{W}_1$*). Moreover, interchanging the components of the decomposition in (2) amounts to taking the dual* $\hat{\mathcal{T}}^2$ *of* $\mathcal{T}^2$ *in (1).* J *is the Jacobian of the curve* Γ *which appears in (3) and (4) and interchanging* Λ_1 *and* Λ_2 *in (2) amounts to interchanging* $\mathcal{W}_1$ *and* $\mathcal{W}_2$ *in (3) and taking the transpose of both square diagrams in the incidence diagram in (4).*

Summarizing we have the following commutative diagram, determined by $\mathcal{T}^2$ *(only),*

$$
\begin{tikzcd}
J \arrow[r, "\Lambda_2"] \arrow[d, "\Lambda_1"] \arrow[dr, "\mathbf{2}_J"] & \hat{\mathcal{T}}^2 \arrow[d, "\Lambda_1"] \\
\mathcal{T}^2 \arrow[r, "\Lambda_2"] & J
\end{tikzcd}
\tag{30}
$$

where $\mathbf{2}_J$ *denotes multiplication by 2 in* J *and a* Λ_i *labeling an arrow means that a projection is considered on the quotient torus that is obtained by doubling the sublattice* Λ_i.

Proof

$(3) \to (2)$ Given a genus two curve Γ and a decomposition $\mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2$ of its Weierstraß points, with $\#\mathcal{W}_i = 3$, let $\pi: \Gamma \to \mathbb{P}^1$ be any two-sheeted cover of $\mathbb{P}^1$. It is well known that π has branch points exactly at $\mathcal{W}$; the points in $\mathcal{W}$ as well as their projections under π will be denoted by $W_1, \ldots, W_6$, also $\pi(\mathcal{W}_i)$ will just be written as $\mathcal{W}_i$. If $\mathbb{P}^1$ is covered with connected open subsets U_1 and U_2 for which $\mathcal{W}_i \subset U_i$ and $U_1 \cap U_2 \cap \mathcal{W} = \emptyset$ then $H_1(\Gamma, \mathbb{Z})$ decomposes as $H_1 \oplus H_2$ where H_1 and H_2 are defined as

$$H_i = \{\gamma \in H_1(\Gamma, \mathbb{Z}) \mid \pi_* \gamma \in H_1(U_i \setminus \mathcal{W}_i, \mathbb{Z})\}.$$

Among the cycles in H_i there are those which come from simple closed curves in $U_i \setminus \mathcal{W}_i$ encercling two points in $\mathcal{W}_i$ and these generate H_i. Since any (different) of these intersect (once) the restriction $\sharp(\cdot)_{H_i}$ is non-degenerate, hence leads (upon using (29)) to a non-degenerate simple decomposition $\Lambda = \Lambda_1 \oplus \Lambda_2$ for the period lattice. Thus $\mathbb{C}^2/\Lambda$ and $\Lambda = \Lambda_1 \oplus \Lambda_2$ provide the corresponding data.

We now show that the constructed data only depend (up to isomorphism) on the isomorphism class of the data $\Gamma, \mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2$. Let $\sigma: \Gamma \to \Gamma$ be an automorphism which permutes the Weierstraß points (such an automorphism only exists for special curves Γ). Then σ extends linearly to $\mathrm{Jac}(\Gamma) \cong \mathbb{C}^2/\Lambda$, hence also to the lattice Λ, giving a new decomposition $\Lambda = \sigma\Lambda_1 \oplus \sigma\Lambda_2$. The lattice $\sigma\Lambda_i$ contains the periods corresponding to the points $\sigma\mathcal{W}_i$ (w.r.t. the same basis of holomorphic differential forms), hence $\Lambda = \sigma\Lambda_1 \oplus \sigma\Lambda_2$ corresponds to the decomposition $\mathcal{W} = \sigma\mathcal{W}_1 \cup \sigma\mathcal{W}_2$.

(2) $\rightarrow$ (3) By the classical Torelli Theorem, Γ can be reconstructed from its Jacobian, actually in dimension two, Γ is isomorphic to the theta divisor of $\mathrm{Jac}(\Gamma)$. The lattice $\Lambda \subset \mathbb{C}^2$ is the period lattice of Γ with respect to some basis $\vec{\omega} = \{\omega_1, \omega_2\}$ of holomorphic differentials on Γ, which determines an isomorphism $\phi\colon \Lambda \rightarrow H_1(\Gamma, \mathbb{Z})$, which in turn leads to a decomposition $H_1(\Gamma, \mathbb{Z}) = H_1 \oplus H_2$ upon defining $H_i = \phi(\Lambda_i)$.

If we denote by $\mathcal{W}$ the set of Weierstraß points of Γ and by $\pi\colon \Gamma \rightarrow \mathbb{P}^1$ any two-sheeted cover as above, then H_i has generators $\lambda_{i1}, \lambda_{i2}$ for which $\pi_*\lambda_{ij}$ is a simple closed curve in $\mathbb{P}^1 \setminus \mathcal{W}$, encircling an even number of branch points W_i, which reduces to two in this case (there are only six points W_i and encircling four points amounts to the same as encircling the other two points). Since the decomposition is non-degenerate, $\pi_*\lambda_{i1}$ and $\pi_*\lambda_{i2}$ encircle a common point, so we may take

$$\mathcal{W}_i = \pi^{-1}\{\text{points in } \mathcal{W} \text{ encircled by } \pi_*\lambda_{i1} \text{ or } \pi_*\lambda_{i2}\}.$$

Then $\#\mathcal{W}_1 = \#\mathcal{W}_2 = 3$ and it is easy to see that $\mathcal{W}_1 \cap \mathcal{W}_2 = \emptyset$.

We show again that the constructed data are independent of the choice of the base $\{\omega_1, \omega_2\}$ and are well-defined up to isomorphism. To do this remark first that when the choice of base $\vec{\omega} = {}^t(\omega_1, \omega_2)$ is not unique, say $\vec{\omega}'$ is another base producing Λ, then $\vec{\omega} = A\vec{\omega}'$ for some $A \in GL(2, \mathbb{C})$, hence

$$\oint_\gamma \vec{\omega} = A \oint_\gamma \vec{\omega}'$$

for any $\gamma \in H_1(\Gamma, \mathbb{Z})$. We find that $\Lambda = A\Lambda$, i.e., Λ has a non-trivial symmetry group. Then $\mathrm{Jac}(\Gamma) = \mathbb{C}^2/\Lambda$ has a non-trivial automorphism group and the data $(\mathbb{C}^2/\Lambda, \Lambda = \Lambda_1 \oplus \Lambda_2)$ and $(\mathbb{C}^2/\Lambda, \Lambda = A\Lambda_1 \oplus A\Lambda_2)$ are isomorphic. Thus it suffices to show that the constructed data are well-defined up to isomorphism. This follows (as in the first part of the proof) at once from the property that if $\mathrm{Jac}(\Gamma)$ has a non-trivial automorphism σ, then it is induced by an automorphism on Γ. To see this property (which is particular for the case in which the genus of Γ is 2) let Θ be a generic translate of the Riemann theta divisor passing through the origin O of $\mathrm{Jac}(\Gamma)$. Then $\sigma(\Theta)$ is another translate passing through O (since every curve in $\mathrm{Jac}(\Gamma)$ which is isomorphic to Γ is a translate of Θ) hence composing σ with this translate determines an automorphism of Γ. This shows the constructed data are well-defined.

(2) $\rightarrow$ (1) Given $J = \mathbb{C}^2/\Lambda$ and $\Lambda = \Lambda_1 \oplus \Lambda_2$ we form the complex torus

$$\mathcal{T}^2 = \mathbb{C}^2/\Lambda' \text{ with } \Lambda' = \frac{1}{2}\Lambda_1 \oplus \Lambda_2,$$

(i.e., the first lattice is doubled in both directions) and equip this torus with the polarization induced by the principal polarization on J. We claim that $\mathcal{T}^2$ is a $(1,4)$-polarized Abelian surface which belongs to $\tilde{\mathcal{A}}_{(1,4)}$. To show this, first notice that the cycles $\{\lambda_{11}, \lambda_{21}, \lambda_{12}, \lambda_{22}\}$ introduced above, form a symplectic base for $H_1(\Gamma, \mathbb{Z})$, i.e., $\sharp(\lambda_{1i} \cdot \lambda_{2i}) = 0$, $\sharp(\lambda_{i1} \cdot \lambda_{i2}) = 1$, hence these cycles lead to a period matrix of the form (see [GH])

$$\begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & b & c \end{pmatrix}$$

satisfying the Riemann conditions. Since H_1 is spanned by λ_{11} and λ_{12} (which correspond to the first and third columns of this matrix) Λ' has in terms of slightly different coordinates the period matrix

$$\begin{pmatrix} 1 & 0 & a & 2b \\ 0 & 4 & 2b & 4c \end{pmatrix}$$

which leads immediately to the result that $\mathcal{T}^2$ is a $(1,4)$-polarized Abelian surface, 4:1 isogeneous to J (remark that the right block of this matrix is positive definite). Since the original $J = \mathbb{C}^2/\Lambda$ is the canonical Jacobian of $\mathcal{T}^2$, we are in the generic case of Section 2 which implies $\mathcal{T}^2 \in \tilde{\mathcal{A}}_{(1,4)}$.

Dually the surface is (up to isomorphism) also constructed by taking

$$\mathcal{T}^2 = \mathbb{C}^2/\Lambda'' \text{ with } \Lambda'' = \Lambda_1 \oplus 2\Lambda_2,$$

but this decomposition induces a 4:1 isogeny from J to (this) $\mathcal{T}^2$.

To show that the correspondence is well-defined, remark that

$$(\mathbb{C}^2/\Lambda,\, \Lambda = \Lambda_1 \oplus \Lambda_2) \cong (\mathbb{C}^2/\Lambda,\, \Lambda = \Lambda_1' \oplus \Lambda_2')$$

implies

$$\mathbb{C}^2 \Big/ \left(\frac{1}{2}\Lambda_1 \oplus \Lambda_2\right) \cong \mathbb{C}^2 \Big/ \left(\frac{1}{2}\Lambda_1' \oplus \Lambda_2'\right) \quad \text{and} \quad \mathbb{C}^2 \Big/ (\Lambda_1 \oplus 2\Lambda_2) \cong \mathbb{C}^2 \Big/ (\Lambda_1' \oplus 2\Lambda_2')$$

the last two isomorphisms being isomorphism of polarized Abelian surfaces.

(1) $\rightarrow$ (2) For given $\mathcal{T}^2 \in \mathcal{A}^0_{(1,4)}$, let J be its canonical Jacobian $J(\mathcal{T}^2)$. Then $\mathcal{T}^2 \rightarrow J$ is part of the isogeny $\mathbf{2}_J\colon J \rightarrow J$ hence there is a unique complementary isogeny $J \rightarrow \mathcal{T}^2$ with kernel $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Writing J as $J = \mathbb{C}^2/\Lambda$, the latter isogeny induces an injective lattice homomorphism $\phi\colon \Lambda \rightarrow \Lambda$ whose cokernel is isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Then ϕ determines a unique decomposition $\Lambda_1 \oplus \Lambda_2$ of Λ for which $\phi_{|\Lambda_2}$ is an isomorphism and $\phi_{|\Lambda_1}$ is multiplication by 2. We have seen that such a decomposition is simple. It is also non-degenerate, since otherwise $\mathcal{T}^2$ would not have an induced $(1,4)$-polarization (see Remark 1 below).

Observe that in the exceptional case that $\mathcal{T}^2 \rightarrow J$ is another part of the isogeny $\mathbf{2}_J$, the two isogenies combine to an automorphism of J, leading to isomorphic data in (3).

(3) $\leftrightarrow$ (4) This is classical (see [Hu]); we prove it as follows. Given a decomposition of $\mathcal{W}$, say $\mathcal{W} = \{W_1, W_2, W_3\} \cup \{W_4, W_5, W_6\}$ the corresponding incidence diagram is taken as

$$\begin{matrix} W_{11} & W_{12} & W_{23} & W_{13} \\ W_{45} & W_{36} & W_{16} & W_{26} \\ W_{46} & W_{35} & W_{15} & W_{25} \\ W_{56} & W_{34} & W_{14} & W_{24} \end{matrix} \qquad \begin{matrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{23} & \Gamma_{13} \\ \Gamma_{45} & \Gamma_{36} & \Gamma_{16} & \Gamma_{26} \\ \Gamma_{46} & \Gamma_{35} & \Gamma_{15} & \Gamma_{25} \\ \Gamma_{56} & \Gamma_{34} & \Gamma_{14} & \Gamma_{24} \end{matrix}$$

and obviously the decomposition of $\mathcal{W}$ is reconstructed from it at once. To show that every incidence diagram is of this form, remark at first that we have the freedom to permute the rows as well as the columns, so that we can put $W_{11} = \ldots = W_{66}$ in the upper left corner. The curves Γ_{ij} this point W_{11} belongs to are the entries in the first row and the first column (except Γ_{11}) of the square diagram on the right. If the origin belongs to $\Gamma_{ij} \cap \Gamma jk$, $(j \neq k)$, then it also belongs to Γ_{ik}. Then Γ_{11} is easily identified as the image of the map $\Gamma \rightarrow \mathrm{Jac}(\Gamma)$ defined by

$$P \mapsto \int_{W_i}^{P} \vec{\omega} + \int_{W_j}^{W_k} \vec{\omega} \pmod{\Lambda},$$

and the other three curves are Γ_{lm}, Γ_{mn} and Γln with $\{i,j,k,l,m,n\} = \{1,2,3,4,5,6\}$. Hence the incidence tabel takes the above form from which the decomposition of $\mathcal{W}$ can be read off.

If the curve has non-trivial automorphisms, we define diagrams which correspond to such automorphisms as being isomorphic, so as to obtain the equivalence (3) $\leftrightarrow$ (4) at the level of isomorphism classes.

Finally we concentrate on the dual $\hat{\mathcal{T}}^2$ of $\mathcal{T}^2$ and its relation with the canonical Jacobian of $\mathcal{T}^2$. At first recall from [GH] that the period matrices of $\mathcal{T}^2$ and $\tilde{\mathcal{T}}^2$ relate as

$$\mathcal{T}^2 \sim \begin{pmatrix} 1 & 0 & a & 2b \\ 0 & 4 & 2b & 4c \end{pmatrix} \qquad \hat{\mathcal{T}}^2 \sim \begin{pmatrix} 4 & 0 & 4a & 2b \\ 0 & 1 & 2b & c \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & c & 2b \\ 0 & 4 & 2b & 4a \end{pmatrix}$$

showing that $\hat{\mathcal{T}}^2$ is constructed from J by taking $\Lambda_1 \oplus \frac{1}{2}\Lambda_2$ instead of taking $\frac{1}{2}\Lambda_1 \oplus \Lambda_2$ when constructing $\mathcal{T}^2$ from J. It follows that the isogeny $\mathbf{2}_J$ factorizes via $\hat{\mathcal{T}}^2$ as well and that taking the dual of $\mathcal{T}^2$ corresponds to interchanging the components of the decomposition of Λ. This finishes the proof of the theorem. ∎

Remarks

1) If in (2) above one considers simple degenerate decompositions (instead of non-degenerate) then the decomposition in (3) is altered into $\mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2 \cup \mathcal{W}_3$, $\#\mathcal{W}_i = 2$ and the order of the components in the decomposition of $\mathcal{W}$ is now irrelevant. The corresponding object in (1) is then a Jacobi surface (different from the one in (2)) from which the original Jacobi surface (or the curve) cannot be reconstructed.

2) Since $\binom{6}{3} = 20$, there are 20 different incidence diagrams and 20 possible decompositions of the isogeny $\mathbf{2}_J \colon J \to J$, some of which are isomorphic if and only if J (hence Γ) has a non-trivial automorphism group (i.e., different from $\mathbb{Z}_2$). It follows from the above theorem that the 20 intermediate Abelian surfaces appear in 10 groups of dual pairs.

3) Let $\mathcal{C}^{(2)}$ denote the moduli space of all smooth curves of genus two. Then we have the following isomorphisms

$$\begin{aligned} \tilde{\mathcal{A}}_{(1,4)} &\cong \left\{(\{W_1,W_2,W_3\},\{W_4,W_5,W_6\}) \mid W_i \in \mathbb{P}^1,\, i \neq j \Rightarrow W_i \neq W_j\right\} \Big/ \operatorname{mod} \mathbb{P}GL(2,\mathbb{C}), \\ \mathcal{C}^{(2)} &\cong \left\{\{W_1,W_2,W_3,W_4,W_5,W_6\} \mid W_i \in \mathbb{P}^1,\, i \neq j \Rightarrow W_i \neq W_j\right\} \Big/ \operatorname{mod} \mathbb{P}GL(2,\mathbb{C}); \end{aligned}$$

and both spaces are related by an obvious unramified covering projection $\tilde{\mathcal{A}}_{(1,4)} \to \mathcal{C}^{(2)}$. We have seen that $\tilde{\mathcal{A}}_{(1,4)}$ has a natural structure of an affine variety which is compactified in a natural way into its projective closure, which is the (singular) algebraic variety $\mathcal{M}^3$. At the other hand, $\mathcal{C}^{(2)}$ has also a natural compactification (the Mumford-Deligne compactification). It would be interesting to figure out how both compactifications are related.

4) Among the different ways to define (and characterize) the canonical Jacobian $J(\mathcal{T}^2)$ of $\mathcal{T}^2$, here is a final one. It is that $J = J(\mathcal{T}^2)$ is the only Jacobian for which the diagram

$$\begin{tikzcd} \mathcal{T}^2 \arrow[d, "4:1"] \arrow[dr, "\mathbf{2}_T"] & \\ J \arrow[r, "4:1"] & \mathcal{T}^2 \end{tikzcd}$$

commutes ($\mathbf{2}_T$ is multiplication by 2 on $\mathcal{T}^2$). The proof is easy using the ideas of the above proof. Observe that this diagram is (30) with $\mathcal{T}^2$ and J interchanged; we could drop a superfluous triangle since $\hat{J} = J$.

6. The relation with the canonical Jacobian made explicit

We have shown in Section 5 that there is associated to an Abelian surface of type $(1,4)$ the Jacobi surface of a genus two curve Γ and some additional data. Also we have seen (in Section 3) that these Abelian surfaces appear as invariant surfaces of the Hamiltonian vector field defined by one of the potentials $V_{\alpha\beta}$. This allows us to make this relation very explicit (in two different ways) and to calculate precisely the locus S in $\mathbb{P}^3$ for which the associated quartic fails to be a Kummer surface (and hence the associated $(1,4)$-polarized Abelian surface fails to be birational to an octic). We know of no direct method (i.e., without using integrable systems) to do this. We refer to [Bu] for an alternative approach, under current investigation, which uses another integrable system (some geodesic flow on $SO(4)$).

6.1. An embedding of the Abelian surfaces in $\mathbb{P}^{15}$

Our calculations rely on the explicit construction of an embedding for $\mathcal{T}^2$ in projective space, which is found by using the Laurent solutions to the differential equations (12). Since we know that the potential $V_{\alpha\beta}$ is a.c.i. (for $\alpha \neq \beta$), the vector field X_H has a coherent tree of Laurent solutions (see [AvM1]), in particular it has Laurent solutions depending on $\dim \mathbb{R}^4 - 1 = 3$ free parameters (*principal balances*). Moreover, since the divisor $\mathcal{D}_{fg}$ to be adjoined to a (generic) invariant manifold $\mathcal{A}_{fg}$ is irreducible, there is only one such family. Also q_1, q_2 and $q = q_1p_2 - q_2p_1$ have a simple pole along $\mathcal{D}_{fg}$ since their squares descend to $\mathrm{Jac}(\Gamma)$ with a double pole along (some translate of) its theta divisor. With this information the principal balance is given by

$$
\begin{aligned}
q_1 &= \frac{1}{t}\left[a + \frac{2}{3}((1+a^2-b^2)\alpha + 2ab^2\beta)t^2 + bct^3 + \mathcal{O}(t^4)\right],\\
q_2 &= \frac{1}{t}\left[b + \frac{2}{3}((1+b^2-a^2)\beta + 2ba^2\alpha)t^2 - act^3 + \mathcal{O}(t^4)\right],
\end{aligned}
\tag{31}
$$

where $2a^2 + 2b^2 + 1 = 0$; the series for p_1 and p_2 are found by differentiation. Using the Laurent solutions it is easy to find an embedding of $\mathcal{T}^2_{fg}$ in projective space: since $2\mathcal{D}_{fg}$ induces a polarization of type $(2,8)$, it is very ample and this can be done using the sixteen functions with a double pole along $\mathcal{D}_{fg}$, to wit,

$$
\begin{aligned}
z_0 &= 1, & z_8 &= q_2^2,\\
z_1 &= q_1, & z_9 &= q_1q,\\
z_2 &= q_2, & z_{10} &= q_2q,\\
z_3 &= q = q_1p_2 - q_2p_1, & z_{11} &= (q_1^2+q_2^2)q + \alpha q_1p_2 + \beta q_2p_1,\\
z_4 &= p_1, & z_{12} &= \{q_1, q\},\\
z_5 &= p_2, & z_{13} &= \{q_2, q\},\\
z_6 &= q_1^2, & z_{14} &= 2q_1q_2(q_1^2+q_2^2) + p_1p_2,\\
z_7 &= q_1q_2, & z_{15} &= q^2,
\end{aligned}
$$

where $\{f_1, f_2\} = \dot{f}_1 f_2 - f_1\dot{f}_2$, the Wronskian of f_2 and f_1.

6.2. Abelian surfaces of type (1,4) as quotients of their canonical Jacobians

A first way to compute the correspondence between the data is to use the cover $J \to \mathcal{T}^2$; recall from Section 5 that given $\mathcal{T}^2 \in \tilde{\mathcal{A}}_{(1,4)}$ there is a unique Jacobian $J = J(\mathcal{T}^2)$ such that

$$\begin{array}{ccc} J & & \\ \Big\downarrow{\scriptstyle p_1} & \overset{\mathbf{2}_J}{\searrow} & \\ \mathcal{T}^2 & \xrightarrow{p_2} & J \end{array}$$

yields a factorization of the map $\mathbf{2}_J$ (multiplication by 2). This implies the existence of a singular divisor in $\mathcal{T}^2$ whose components are birational equivalent to $\Gamma = \Gamma(\mathcal{T}^2)$ as is shown in the following proposition.

Proposition 5 *The image $p_1(\mathcal{K})$ of Kummer's 16_6 configuration $\mathcal{K}$ consists of four curves, all passing through the half periods of $\mathcal{T}^2$; these points are the images of the sixteen points in the configuration and each of the four image curves has an ordinary three-fold point at one of these points, with tangents at this point, which are different from the tangents to the other curves. Each curve is birational equivalent to Γ and induces a $(1,4)$-polarization on $\mathcal{T}^2$. The image $p_2(p_1(\mathcal{K}))$ is one single curve, birational equivalent to Γ with an ordinary six-fold point.*

Proof

The map p_1 identifies all half-periods which appear in a row in the first square diagram of the incidence diagram which corresponds to $\mathcal{T}^2$. Therefore p_1 also identifies the curves which appear in a row in the second square diagram of this incidence diagram and we obtain four curves passing through the four image points, every curve having a three-fold point at the image of the three points in the same row (but not the same column) of the first square diagram. Since $\mathcal{K}$ induces a $(16,16)$-polarization on J, $p_1(\mathcal{K})$ induces a $(4,16)$-polarization on $\mathcal{T}^2$, hence each component induces a $(1,4)$-polarization. The virtual genus of each component is thus five, and since each is obviously birational to Γ via p_1, the threefold point must be ordinary and there are no other singular points.

The intersection of two of these components is the self-intersection of one of them (since they are translates of each other), hence is by (5) equal to $2(5-1) = 8$; at the other hand, since each passes through the three-fold point of the other and since they have two simple points in common, this gives already $3+3+1+1 = 8$ so all tangents must be different and there are no other intersection points. The fact that $p_2(p_1(\mathcal{K}))$ has an ordinary six-fold point and is birational equivalent to Γ is shown in a similar way. ∎

The image $\mathbf{2}_J(\Theta)$ is a divisor Δ with a six-fold point, first studied in [V1] (where it was an essential ingredient in the construction of linearizing variables for integrable systems) and $p_1(\mathcal{K})$ is nothing but $p_2^*\Delta$. We have also shown there that this divisor is the zero locus of the leading term in the equation of the Kummer surface of J (when normalised as in the algorithm in Section 2.2).

To apply this in the present case, we use the leading term (18) of the equation of the Kummer surface of $J(\mathcal{T}^2_{fg})$ (which is expressed in terms of the original variables), and investigate its zero locus, i.e.,

$$(q_1^2 + q_2^2 + \alpha + \beta)^2 - 4(\alpha\beta + \beta q_1^2 + \alpha q_2^2) = 0.$$

This factorizes completely as

$$\prod_{\epsilon_i = \pm 1} \left[q_2 - \epsilon_1 \sqrt{\alpha - \beta} - \epsilon_2 i q_1 \right] = 0.$$

reflecting the fact that $p_2^*\Delta$ is reducible. In order to find an equation for $\Gamma(\mathcal{T}_{fg}^2)$, let $q_2 = \epsilon_1\sqrt{\alpha-\beta}+\epsilon_2 i q_1$ in the equations for $\mathcal{A}_{fg}$. Eliminating p_2 one finds an equation for the curve

$$\Delta_{\epsilon_1\epsilon_2}\colon p_1^2 Q(q_1)(q_1-\epsilon_1\epsilon_2 i\sqrt{\alpha-\beta})q_1+P^2(q_1)=0,$$

where

$$Q(x)=\epsilon_1\epsilon_2 i(\alpha-\beta)^{3/2}x^3+(\alpha-\beta)(2\alpha-\beta)x^2+\epsilon_1\epsilon_2 i\sqrt{\alpha-\beta}(h+\alpha(\beta-\alpha))x-\frac{f}{2},$$

P is some polynomial of degree 3. This curve is clearly isomorphic to the curve

$$z^2=x(x-i\epsilon_1\epsilon_2\sqrt{\alpha-\beta})Q(x). \tag{32}$$

In order to decide to which decomposition of the Weierstraß points this corresponds, let $P_1,\dots,P_4$ be the following points in $\mathbb{P}^{15}$

$$\begin{aligned}
P_1&=(0:\cdots:0:-i\sqrt{\alpha-\beta}:-\sqrt{\alpha-\beta}:1:+i(\alpha-\beta)),\\
P_2&=(0:\cdots:0:+i\sqrt{\alpha-\beta}:+\sqrt{\alpha-\beta}:1:+i(\alpha-\beta)),\\
P_3&=(0:\cdots:0:+i\sqrt{\alpha-\beta}:-\sqrt{\alpha-\beta}:1:-i(\alpha-\beta)),\\
P_4&=(0:\cdots:0:-i\sqrt{\alpha-\beta}:+\sqrt{\alpha-\beta}:1:-i(\alpha-\beta)),
\end{aligned}$$

and let q_δ denote the three roots of $Q(x)$. Then it is easily checked by picking local parameters around the points at infinity of $\Delta_{\epsilon_1\epsilon_2}$ that the incidence relation of the P_i on the $\Delta_{\epsilon_1\epsilon_2}$ is given by the following table:

	$q_1\to 0$	$q_1\to\infty$	$q_1\to q_\delta$	$q_1\to\epsilon_1\epsilon_2 i\sqrt{\alpha-\beta}$
$\Delta_{+1,+1}$	P_1	P_4	$3P_3$	P_2
$\Delta_{-1,+1}$	P_2	P_3	$3P_4$	P_1
$\Delta_{+1,-1}$	P_3	P_2	$3P_1$	P_4
$\Delta_{-1,-1}$	P_4	P_1	$3P_2$	P_3

Table 4

The table is in agreement with the fact that each curve has a three-fold point and passes through the other singularities. Moreover it shows that the three points q_δ were identified under the map p_1 when going from J to $\mathcal{T}^2$, hence these form the subset $\mathcal{W}_1$ in Theorem 4 and $\mathcal{W}_2=\{0,\infty,\epsilon_1\epsilon_2 i\sqrt{\alpha-\beta}\}$

If we substitute

$$x\mapsto\frac{x+\alpha}{\sqrt{\alpha-\beta}}i.$$

in the equation (33) for the curves $\delta_{\epsilon_1\epsilon_2}$ then we find the equation (21),

$$y^2=(x+\alpha)(x+\beta)\left(x^3+(\alpha+\beta)x^2+(\alpha\beta-h)x+\frac{\beta f-\alpha g}{2(\alpha-\beta)}\right). \tag{33}$$

Then the decomposition of $\mathcal{W}$ is given as follows: $\mathcal{W}_1$ contains the roots of $x^3+(\alpha+\beta)x^2+(\alpha\beta-h)x+(\beta f-\alpha g)/(2\alpha-2\beta)$, and $\mathcal{W}_2=\{\infty,-\alpha,-\beta\}$.

6.3. Abelian surfaces of type $(1,4)$ as covers of their canonical Jacobians

An alternative way to compute the data corresponding to $\mathcal{T}^2_{fg}$ is by using the cover $\mathcal{T}^2 \to J$. First we calculate an equation for $\mathcal{D}_{fg}$ by substituting (31) in the invariants. Eliminating one of the free parameters from the resulting equations yields the following equations defining a curve:

$$\mathcal{D}_{fg}\colon \begin{cases} -\dfrac{9}{4}c^2 = 16(\beta-\alpha)^3a^6 + 8(\beta-\alpha)^2(\beta-2\alpha)a^4 - 4(\beta-\alpha)(h+\alpha(\beta-\alpha))a^2 - f, \\ -1 = 2a^2 + 2b^2. \end{cases} \tag{34}$$

Put

$$x = i\sqrt{2(\alpha-\beta)}a, \qquad y = \sqrt{2(\alpha-\beta)}b,$$

to find that this non-singular curve is isomorphic to the curve

$$\mathcal{C}\colon \begin{cases} z^2 = x^6 + (\beta-2\alpha)x^4 - (h+\alpha(\beta-\alpha))x^2 - \dfrac{f}{2}, \\ y^2 = x^2 + \beta - \alpha. \end{cases} \tag{35}$$

To check that the genus of $\mathcal{D}_{fg}$ equals 5 (as we saw in Section 3), let $\mathcal{C}'$ denote the curve

$$\mathcal{C}'\colon z^2 = x^6 + (\beta-2\alpha)x^4 - (h+\alpha(\beta-\alpha))x^2 - \frac{f}{2},$$

which has genus two. Then the obvious map $\pi\colon \mathcal{C} \to \mathcal{C}'$ is a 2:1 covering map with four ramification points (the points where $y=0$). By Riemann-Hurewicz,

$$\chi(\mathcal{D}_{fg}) = 2\chi(\mathcal{C}') - \text{ramification}(\pi), \tag{36}$$

it follows that $g(\mathcal{D}_{fg}) = 5$.

Letting $t = x^2 - \alpha$, (36) is obviously equivalent to

$$\begin{cases} z^2 = t^3 + t^2(\alpha+\beta) + t(\alpha\beta - h) + \dfrac{\beta f - \alpha g}{2(\alpha-\beta)}, \\ x^2 = \alpha + t, \\ y^2 = \beta + t \end{cases}$$

where we used $f - g = 2(\beta-\alpha)h$ in the first equation to write it in a symmetric form. Define now $u = xyz$ and find that $\mathcal{D}_{fg}$ is expressed as a 4:1 unramified cover

$$\begin{cases} u^2 = (t+\alpha)(t+\beta)\left(t^3 + t^2(\alpha+\beta) + t(\alpha\beta-h) + \dfrac{\beta f - \alpha g}{2(\alpha-\beta)}\right), \\ x^2 = \alpha + t, \\ y^2 = \beta + t \end{cases}$$

of the hyperelliptic curve given by

$$z^2 = (t+\alpha)(t+\beta)\left(t^3 + t^2(\alpha+\beta) + t(\alpha\beta-h) + \frac{\beta f - \alpha g}{2(\alpha-\beta)}\right),$$

which we found in (21) and (34). To see this, remark that if $u^2 = f(t)$ is an equation of any hyperelliptic curve Γ and $f(t_1) = f(t_2) = f(t_3) = 0$ (i.e., t_1, t_2 and t_3 correspond to Weierstraß points), then the curve

$$u^2 = f(t), \qquad x^2 = \frac{t - t_1}{t - t_2}, \qquad y^2 = \frac{t - t_2}{t - t_3},$$

is a 4:1 cover of Γ and has genus 5; in our case $\{t_1, t_2, t_3\} = \{\infty, -\alpha, -\beta\}$. When this 4:1 cover is extended to the cover $\mathcal{T}^2 \to \mathrm{Jac}(\Gamma)$ the half-periods on $\mathcal{T}^2$ corresponding to $\{-\alpha, -\beta, \infty\}$ are identified with the origin, hence $\mathcal{W}_2 = \{-\alpha, -\beta, \infty\}$ and $\mathcal{W}_1$ consists of the other three Weierstraß points, in agreement with our previous calculation.

6.4. The exceptional locus $S \subset \mathbb{P}^3$

Suppose that $(\mathcal{T}^2, \mathcal{L}) \in \tilde{\mathcal{A}}_{(1,4)}$ and let the surface be represented by a surface $\mathcal{A}_{(\alpha,\beta,f,g)}$, for some $\alpha \neq \beta$ (using (28)). Then the curve $\Gamma(\mathcal{T}^2)$ corresponding to it under the basic bijection explained in Section 5 must be smooth. Since we know from Section 6.2 (or equivalently 6.3) that an equation for $\Gamma(\mathcal{T}^2)$ is given by

$$y^2 = (x + \alpha)(x + \beta)P_3(x), \qquad P_3(x) = x^3 + (\alpha + \beta)x^2 + (\alpha\beta - h)x + \frac{\beta f - \alpha g}{2(\alpha - \beta)}. \tag{37}$$

we conclude that $\mathrm{disc}(P_3(x)) \neq 0$ and $P_3(-\alpha) \neq 0$, $P_3(-\beta) \neq 0$, the last condition meaning just that $f \neq 0$ and $g \neq 0$. Conversely, both conditions together are sufficient to guaranty that the curve is smooth and the corresponding Abelian surface is in $\tilde{\mathcal{A}}_{(1,4)}$. In order to state this result in terms of the coordinates μ_i for $\mathbb{P}^3$, use (28) to rewrite (38) in the simple form $y^2 = x(x-1)P_3^\mu(x)$ where

$$P_3^\mu(x) = 4\mu_2 x^3 - (\mu_0 + 2\mu_1 + 6\mu_2 + 2\mu_3)x^2 + (\mu_0 - 2\mu_1 + 2\mu_2 + 6\mu_3)x - 4\mu_3,$$

(x and y are slightly rescaled); in this representation $\mathcal{W}_2 = \{0, 1, \infty\}$ and $\mathcal{W}_1$ contains the roots of $P_3^\mu(x)$. The condition for $(\mu_0 : \mu_1 : \mu_2 : \mu_3)$ to correspond to a surface in $\tilde{\mathcal{A}}_{(1,4)}$ is now that $\mu_1\mu_2\mu_3 \neq 0$ and $\mathrm{disc}(P_3^\mu(x)) \neq 0$. It shows that the locus S' is given by the four divisors $\mu_1\mu_2\mu_3 = 0$ and $\mathrm{disc}(P_3^\mu(x)) = 0$ and the exceptional locus S is found immediately from it by substituting λ_i^2 for μ_i in these equations. (These equations for S can in principle be found purely algebraic, but the calculations are very tedious and some cases are easily overlooked. In fact [BLS] claim (without proof) in their paper that the only condition is $\mu_1\mu_2\mu_3 \neq 0$, thereby overlooking the more subtle condition $\mathrm{disc}(P_3^\mu(x)) \neq 0$). Combining this with Theorem 1 we have shown the following theorem.

Theorem 6 *The surface $\mathcal{A}_{(\alpha,\beta,f,g)}$ is an affine part $\mathcal{T}^2 \backslash \mathcal{D}$ of an Abelian surface $(\mathcal{T}^2, [D]) \in \tilde{\mathcal{A}}_{(1,4)}$ if and only if $\alpha \neq \beta$, $f \neq 0, g \neq 0$ and $\mathrm{disc}(P_3(x)) \neq 0$. Equivalently $(\mu_0 : \mu_1 : \mu_2 : \mu_3) \in \mathbb{P}^3$ are moduli coming from the birational map[3] $\phi_{\mathcal{L}} : \mathcal{T}^2 \to \mathbb{P}^3$ with $(\mathcal{T}^2, \mathcal{L}) \in \tilde{\mathcal{A}}_{(1,4)}$ if and only if $\mu_1\mu_2\mu_3 \neq 0$ and $\mathrm{disc}(P_3^\mu(x)) \neq 0$. The curve $\Gamma(\mathcal{T}^2)$ corresponding to the canonical Jacobian of $\mathcal{T}^2$ is then written as*

$$y^2 = x(x-1)\left(4\mu_2 x^3 - (\mu_0 + 2\mu_1 + 6\mu_2 + 2\mu_3)x^2 + (\mu_0 - 2\mu_1 + 2\mu_2 + 6\mu_3)x - 4\mu_3\right),$$

[3] recall that $\mu_i = \lambda_i^2$, where λ_i are taken from (7)

when the coordinates x for $\mathbb{P}^1$ is taken such that $\mathcal{W}_2 = \{0, 1, \infty\}$. Conversely the equation of the octic (7) is written down at once when giving the equation of the genus two curve and a decomposition $\mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2$ of its set of Weierstraß points: the coefficients of the octic are $\lambda_i = \sqrt{\mu_i}$ where μ_i are essentially the symmetric functions of $\mathcal{W}_2$ when the coordinate x for $\mathbb{P}^1$ is taken such that $\mathcal{W}_2 = \{0, 1, \infty\}$.

Taking also the non-generic case into account, there is an Abelian surface $\mathcal{A}_{(\alpha,\beta,f,g)}$ corresponding to each point in the image $\psi(\mathcal{A}_{(1,4)}) = (\mathcal{M}^3 \setminus D) \cup (C \setminus \{P, Q\})$.

∎

The following important corollary follows at once from this theorem.

Corollary 7 *For any Abelian surface $(\mathcal{T}^2, [\mathcal{D}]) \in \tilde{\mathcal{A}}_{(1,4)}$ the affine variety $\mathcal{T}^2 \setminus \mathcal{D}$ is (isomorphic to) a complete intersection of two quartics in $\mathbb{C}^4$.*

∎

Remarks

1) Recalling the description of $\tilde{\mathcal{A}}_{(1,4)}$ from Remark 5.2 one has the following description of the moduli space $\tilde{\mathcal{A}}_{(1,4)}$:

$$\begin{aligned}\tilde{\mathcal{A}}_{(1,4)} &\cong \left\{(\{W_1, W_2, W_3\}, \{W_4, W_5, W_6\}) \mid W_i \in \mathbb{P}^1,\, i \neq j \Rightarrow W_i \neq W_j\right\} \Big/ \mathrm{mod}\, \mathbb{P}GL(2, \mathbb{C}), \\ &\cong \left\{\{W_4, W_5, W_6\} \mid W_i \in \mathbb{C} \setminus \{0, 1\},\, i \neq j \Rightarrow W_i \neq W_j\right\} \Big/ S_3,\end{aligned}$$

where the action of S_3 consists of permuting $0, 1$ and ∞ in the equation $y^2 = x(x-1)(x-W_4)(x-W_5)(x-W_6)$, i.e., it is generated by replacing x by $1/x$ and $1-x$ in this equation. Obviously the ring of invariants of the symmetric functions of W_4, W_5 and W_6 is just the cone $\mathcal{M}^3$, which explains why $\tilde{\mathcal{A}}_{(1,4)}$ has such a nice structure. Using Table 2, this leads to a geometric interpretation of the "intermediate" moduli space $\mathbb{P}^3 \setminus S'$, namely

$$\mathbb{P}^3 \setminus S' \cong \left\{\{W_4, W_5, W_6\} \mid W_i \in \mathbb{C} \setminus \{0, 1\},\, i \neq j \Rightarrow W_i \neq W_j\right\}.$$

To explain this, remark that taking the base vectors mod 2 in the third column of Table 2 determines an ordering for the 4 half-periods on the canonical Jacobian which correspond to the lattice Λ_2, which in turn induce an ordering in the points in $\mathcal{W}_2$; at the other hand, all elements in the second column of Table 1 are the same mod 2.

2) In the classical literature one defines a *Rosenhain tetrahedron* for a Kummer surface as a tetrahedron in $\mathbb{P}^3$ with singular planes of the surface as faces and singular points of it as vertices. In [Hu] the author shows that the equation for the Kummer surface with respect to a Rosenhain tetrahedron is written as the quartic (15). It then follows from Theorem 6 how to read off from the equation of a Kummer surface with respect to a Rosenhain tetrahedron, an equation for the curve corresponding to this Kummer surface and vice versa. It seems that this result is not known in the classical or recent literature.

7. The central potentials $V_{\alpha\alpha}$

In this final section we concentrate on the potentials $V_{\alpha\alpha}$ which were always excluded up to now. It is interesting to compare the classical linearization of the central potential $V_{\alpha\alpha}$ which uses polar coordinates with the $\alpha = \beta$-limit of the linearization of the perturbed potential $V_{\alpha\beta}$ ($\alpha \neq \beta$): they will be seen to coincide. We will also construct a Lax pair for this limiting case and discuss the geometry of the invariant manifolds of the vector field.

At first, consider for generic values of h, k the invariant surface $\mathcal{A}_{hk}$ defined by

$$\mathcal{A}_{hk}\colon \begin{cases} h = \frac{1}{2}\left(p_1^2 + p_2^2\right) + \left(q_1^2 + q_2^2\right)^2 + \alpha\left(q_1^2 + q_2^2\right), \\ k = q_1 p_2 - q_2 p_1, \end{cases}$$

which in terms of polar coordinates (ρ, θ) becomes

$$\begin{aligned} h &= \frac{1}{2}\left(\dot\rho^2 + \rho^2\dot\theta^2\right) + \rho^4 + \alpha\rho^2, \\ k &= \rho^2\dot\theta, \end{aligned}$$

leading to

$$-\frac{1}{2}\rho^2\dot\rho^2 = \rho^6 + \alpha\rho^4 - h\rho^2 + \frac{k^2}{2}.$$

This suggests setting $\sigma = \rho^2$, yielding

$$-\frac{\dot\sigma^2}{8} = \sigma^3 + \alpha\sigma^2 - h\sigma + \frac{k^2}{2}. \tag{38}$$

Secondly the transformation (19) reduces for $\alpha = \beta$ to

$$\begin{aligned} x_1 + x_2 &= -\left(q_1^2 + q_2^2 + 2\alpha\right), \\ x_1 x_2 &= \alpha^2 + \alpha q_1^2 + \alpha q_2^2, \end{aligned} \tag{39}$$

and (20) becomes

$$\dot x_i^2 = \frac{8(x_i + \alpha)^2\left(x_i^3 + 2\alpha x_i^2 + (\alpha^2 - h)x_i - (h\alpha + f/2)\right)}{(x_1 - x_2)^2} \tag{40}$$

The equivalence of (39) and (41) becomes clear after the simple translation $x_i = x_i + \alpha$ on the curve; indeed (40) becomes

$$\begin{aligned} s_1 + s_2 &= -\left(q_1^2 + q_2^2\right), \\ s_1 s_2 &= 0, \end{aligned}$$

so that only one of the s_i differs from zero, say $0 \neq s_1 = -(q_1^2 + q_2^2) = -s$, (the last equality is a definition), which matches the linearizing variable σ introduced above. In terms of s (41) is reduced to one equation which reads

$$-\frac{\dot s^2}{8} = s^3 + \alpha s^2 - hs + \frac{f}{2},$$

which is exactly (39) since $f = (q_1 p_2 - q_2 p_1)^2 = k^2$.

It is also interesting that the Lax pair gives in the limit $\alpha = \beta$ a Lax pair for the potential $V_{\alpha\alpha}$. The polynomials $u(x), v(x)$ and $w(x)$ are now all divisible by $(x+\alpha)$,

$$
\begin{aligned}
u(x) &= (x+\alpha)\left(x + q_1^2 + q_2^2 + \alpha\right), \\
v(x) &= \frac{1}{\sqrt{2}}(x+\alpha)\left(q_1p_1 + q_2p_2\right), \\
w(x) &= (x+\alpha)\left(x^2 + \left(\alpha - q_1^2 - q_2^2\right)x - \frac{1}{2}\left(p_1^2+p_2^2\right) - \alpha\left(q_1^2+q_2^2\right)\right),
\end{aligned}
$$

which leads to a simpler Lax pair by canceling the factor $(x+\alpha)$.

Finally we describe the affine invariant surfaces for the central potentials $V_{\alpha\alpha}$. These turn out to be $\mathbb{C}^*$-bundles over the elliptic curves (39), as described in the following theorem.

Theorem 8 *For any $k, h \in \mathbb{C}$, let $\mathcal{A}_{hk}$ denote the affine surface defined by*

$$
\mathcal{A}_{hk}\colon \begin{cases} h = \dfrac{1}{2}\left(p_1^2+p_2^2\right) + \left(q_1^2+q_2^2\right)^2 + \alpha\left(q_1^2+q_2^2\right), \\ k = q_1p_2 - q_2p_1. \end{cases} \tag{41}
$$

If $k \neq 0$ then $\mathcal{A}_{hk}$ is a $\mathbb{C}^*$-bundle over the elliptic curve

$$
\mathcal{E}_{hk}\colon -\frac{\tau^2}{2} = \sigma^3 + \alpha\sigma^2 - h\sigma + \frac{k^2}{2}. \tag{42}
$$

Moreover the $\mathbb{C}^*$-action on $\mathcal{A}_{hk}$ is a Hamiltonian action, the Hamiltonian function corresponding to it being the momentum $q_1p_2 - q_2p_1$.

Proof

The linearizing variables, calculated above suggest to consider the map

$$
\begin{aligned}
\xi\colon \quad & \mathbb{C}^4 \quad \to \mathbb{C}^2 \\
& (q_1,q_2,p_1,p_2) \mapsto (\sigma,\tau) = \left(q_1^2+q_2^2, q_1p_1+q_2p_2\right).
\end{aligned}
$$

Our first aim is that the image $\xi(\mathcal{A}_{hk})$ is given by the plane elliptic curve (43). Indeed, one easily obtains for $q_1^2+q_2^2 \neq 0$,

$$
\begin{aligned}
p_1 &= \frac{q_2k - q_1\tau}{q_1^2+q_2^2}, \\
p_2 &= -\frac{q_1k + q_2\tau}{q_1^2+q_2^2},
\end{aligned}
$$

which leads by direct substitution in the first equation of (42) immediately to

$$
-\frac{\tau^2}{2} = \sigma^3 + \alpha\sigma^2 - h\sigma + \frac{k^2}{2}.
$$

For $q_1^2+q_2^2 = 0$, i.e., $q_2 = \pm i q_1$ one gets

$$
\begin{aligned}
h &= \frac{1}{2}(p_1^2+p_2^2), \\
k &= q_1(p_2 \mp i p_1), \\
\tau &= q_1(p_1 \pm i p_2),
\end{aligned}
$$

from which we deduce $\tau = \pm ik$, giving the point $(\sigma, \tau) = (0, \pm ik)$ on $\mathcal{E}_{hk}$, proving the first claim.

Secondly, we determine the fiber $\xi^{-1}(\sigma, \tau)$ over each point on $\mathcal{E}_{hk}$. To do this, observe that the multiplicative group of non-zero complex numbers,

$$\mathbb{C}^* \cong \mathbf{SO}(2, \mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a^2 + b^2 = 1 \right\}$$

acts on $\mathcal{A}_{hk}$ by

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \cdot \begin{pmatrix} q_1 & p_1 \\ q_2 & p_2 \end{pmatrix} = \begin{pmatrix} aq_1 + bq_2 & ap_1 + bp_2 \\ aq_2 - bq_1 & ap_2 - bp_1 \end{pmatrix}$$

and the surjective map ξ is $\mathbb{C}^*$-invariant. It is proved by direct calculation that the action is free, hence each fiber of ξ consists of one or more circles. If $(\sigma, \tau) \in \mathcal{E}_{hk}$ then p_1 and p_2 are determined from q_1 and q_2 (at least if $q_1^2 + q_2^2 \neq 0$), which themselves are determined (up to the action of $\mathbb{C}^*$) by $q_1^2 + q_2^2 = \rho$, so exactly one circle lies over each point (q_1, q_2, p_1, p_2) for which $q_1^2 + q_2^2 \neq 0$; in the special case that $q_1^2 + q_2^2 = 0$, the same is true, since p_1 and p_2 are determined (up to the action of $\mathbb{C}^*$) by $p_1^2 + p_2^2 = 2h$, and q_1, q_2 are uniquely determined from p_1 and p_2. It follows that $\mathcal{A}_{hk}$ is a $\mathbb{C}^*$-bundle over the elliptic curve $\mathcal{E}_{hk}$.

Finally, remark that the Hamiltonian vector field corresponding to the momentum $q_1p_2 - q_2p_1$ is given by

$$\begin{aligned} \dot{q}_1 &= -q_2, & \dot{p}_1 &= -p_2, \\ \dot{q}_2 &= q_1, & \dot{p}_2 &= p_1, \end{aligned}$$

from which it is seen that the complex flow of this vector field is given by the $\mathbb{C}^*$-action, proving the last claim in the theorem. ∎

Let us define (and calculate) the moduli (in $\mathbb{P}^{(1,2,2,3,4)}$) corresponding to an invariant surface $\mathcal{A}_{hk}$ of a central potential for $k \neq 0$ as the limit[4]

$$\lim_{\alpha \to \beta} \tilde{\psi}(\mathcal{T}^2_{(\alpha,\beta,f,g)}), \qquad f = k^2.$$

Then an easy computation shows that this limit exists, is independent of $f \neq 0$, h and $\alpha = \beta$ and moreover is exactly equal to the special point P at the boundary of $\psi(\mathcal{A}_{(1,4)})$ defined in Theorem 3. Namely for $f \to g$ and $\alpha \to \beta$ one finds

$$(\mu_0 \colon \mu_1 \colon \mu_2 \colon \mu_3) = (-4 \colon 1 \colon 0 \colon 1)$$

so that

$$(f_0, f_1, f_2, f_3, f_4) = (-4 \colon 0 \colon 3 \colon -2 \colon 0)$$

hence by weight homogeneity the associated moduli correspond to P. Remark that the point is independent of $\alpha = \beta$ as well as of $f = g$, so the map ψ does not distinguish between any of the invariant surfaces of any central potential $V_{\alpha\alpha}$.

Acknowledgments. I wish to thank M. Adler, J. Bertin, P. Bueken, Ch. Birkenhake, K. Hulek, W. Obbels, A. Perelomov, P. van Moerbeke and D. van Straten for several useful comments and stimulating discussions concerning this paper. The warm hospitality of the Max-Planck-Institut, where the final version of this paper was written, is also greatly acknowledged.

[4] recall that $f - g = 2(\beta - \alpha)h$

8. Appendix: The Schlesinger system, the Garnier system and the quartic potentials V_λ

In this appendix we explain the origin of the quartic potentials V_λ, which were first discovered and studied in the beginning of this century by Garnier in [G]. Our exposition is along the lines of that paper.

At first, consider a linear differential equation of order m with $n+3$ regular singularities, say at the points $t_1, \ldots, t_n, t_{n+1} = 0, t_{n+2} = 1$ in the plane and at infinity (it is convenient to put also $x = t_0$). The most general form of such an equation is given by

$$\frac{dy_k}{dx} = \sum_{h=1}^{m} y_h \sum_{i=1}^{n+2} \frac{A^i_{hk}}{x - t_i} \qquad (h = 1, \ldots, m),$$

the A^i_{hk} being constants. This can be written more compactly in matrix-form as

$$\frac{dy}{dx} = yA \tag{43}$$

upon defining a matrix A with entries

$$A_{hk} = \sum_{i=1}^{n+2} \frac{A^i_{hk}}{x - t_i};$$

it has m independent solutions $y_1(x), \ldots, y_m(x)$ which are multivalued functions of x. Using m fundamental solutions as rows in a matrix, an $m \times m$-matrix Y is formed. When such a matrix solution $Y_1(x)$ is continued analytically around a closed path encircling a singular point t_i, then a new solution $Y_2(x)$ is obtained, which is a matrix whose rows are linear combinations of the rows of $Y_1(x)$, hence there is an associated monodromy matrix M_i defined by

$$Y_2(x) = M_i Y_1(x).$$

In this way, $n+3$ monodromy matrices are obtained and they depend on the position of the poles t_i as well as on the values of the constants A^i_{hk}. One of the basic problems in the classical work about linear differential equations is the following isomonodromic problem:

> *How can one make the coefficients A^i_{hk} dependent on $t_1, \ldots, t_n$ such that the monodromy matrices M_i become independent of $t_1, \ldots, t_n$?*

Schlesinger shows in [S] that the dependence of the matrices $A^i = (A^i_{hk})_{h,k=1,\ldots,m}$ on the t_i is given by the following set of partial differential equations:

$$\begin{aligned} \frac{\partial A^j}{\partial t_i} &= \frac{[A^i, A^j]}{t_j - t_i} \qquad (j \neq i), \\ \sum_{j=1}^{n} \frac{\partial A^j}{\partial t_i} &= 0. \end{aligned} \tag{44}$$

Indeed let Y be a matrix solution of (44),

$$\frac{dY}{dx} = YA,$$

and define

$$\beta_i = Y^{-1}\frac{\partial Y}{\partial t_i} \qquad (i = 0, \ldots, n),$$

in particular define $\beta_0 = A$. Expressing the integrability condition

$$\frac{\partial^2 Y}{\partial t_i \partial t_j} = \frac{\partial^2 Y}{\partial t_j \partial t_i}$$

leads to

$$\frac{\partial \beta_i}{\partial t_j} - \frac{\partial \beta_j}{\partial t_i} = [\beta_i, \beta_j]; \tag{45}$$

moreover it can be shown that β_i is holomorphic, away from $x = t_i$ and $\beta_i + A$ is holomorphic around $x = t_i$. It follows that

$$\beta_i = -\frac{A^i}{x - t_i} + \gamma_i, \tag{46}$$

with γ_i independent of x. Actually, without loss of generality, all γ_i may be supposed to be zero. Expressing (46) in terms of A^i using (47) (with $\gamma_i = 0$) and putting $x = t_j$ leads immediately to Schlesinger's system (45).

From (45), Garnier constructs the so-called simplified system, simply by replacing

$$\begin{aligned} t_i &\to \alpha_i + \epsilon t_i, \qquad (i = 1, \ldots, n) \\ A^i &\to \epsilon^{-1} A^i \end{aligned}$$

and taking the limit $\epsilon \to 0$. The resulting system reads

$$\begin{aligned} \frac{\partial A^j}{\partial t_i} &= \frac{[A^i, A^j]}{\alpha_j - \alpha_i} \qquad (j \neq i) \\ \sum_{j=1}^{n} \frac{\partial A^j}{\partial t_i} &= 0. \end{aligned} \tag{47}$$

If a matrix B is defined as

$$B = Ax(x-1)\prod_{i=1}^{n}(x - \alpha_i),$$

then the entries of B are polynomials in x of degree $n+1$ and the simplified form of (46) for $j = 0$ is given by

$$\frac{\partial B}{\partial t_i} = \frac{[A^i, B]}{x - \alpha_i}. \tag{48}$$

Garnier proves that the spectral curve $\det(B(x) - \lambda z) = 0$ is independent of all t_i and linearizes the flow of the vector field. Observe that the matrices $B = B(x)$ and A^i are related as follows:

$$B(\alpha_i) = A^i \prod_{j \neq i}^{n+2} (\alpha_i - \alpha_j).$$

This shows that the Lax pair coincides with the Lax pair considered by A. Beauville in [Be].

The Lax pair (49) contains a lot of integrable systems. Garnier considers two special cases, which both lead to hyperelliptic curves:

i) $\det(B(x)-\lambda z)=0$ is quadratic in z, i.e., B is a 2×2 matrix: this leads after some suitable normalizations (see [Be]) to what we called the odd master system (see [V1] and [M2]).

ii) $\det(B(x)-\lambda z)=0$ is quadratic in y: then there is no loss of generality in supposing that B has the form

$$B=\begin{pmatrix} x^2+c_{11} & b_{12}x+c_{12} & \cdots & b_{1m}x+c_{1m} \\ b_{21}x+c_{21} & c_{22} & \cdots & b_{2m}x+c_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1}x+c_{m1} & b_{m2}x+c_{m2} & \cdots & c_{mm} \end{pmatrix}$$

Then (49) is written out for $i=1$ as

$$\begin{aligned} \alpha_1(\alpha_1-1)\frac{db_{1k}}{dt_1} &= -(b_{1k}\alpha_1+c_{1k}), \\ \alpha_1(\alpha_1-1)\frac{db_{k1}}{dt_1} &= b_{k1}\alpha_1+c_{k1}, \\ \alpha_1(\alpha_1-1)\frac{dc_{hk}}{dt_1} &= c_{h1}b_{1k}-b_{h1}c_{1k}. \end{aligned} \tag{49}$$

Define ξ_k and η_k by

$$b_{1k}=\xi_k\exp\frac{t_1}{1-\alpha_1},\qquad b_{k1}=\eta_k\exp\frac{t_1}{\alpha_1-1},$$

bring c to its canonical form (supposed here to be diagonal), define $a_i=c_{ii}$ and choose $c_{11}=-\xi_2\eta_2-\cdots-\xi_m\eta_m$. Then (50) reduces to

$$\begin{aligned} \ddot{\xi}_i &= \xi_i(2\sum_{j=2}^m \xi_j\eta_j+a_i), \\ \ddot{\eta}_i &= \eta_i(2\sum_{j=2}^m \xi_j\eta_j+a_i), \end{aligned}$$

an integrable system which is known as the Garnier system. Restricted to the invariant subspace $\xi_i=\eta_i$ it gives exactly Newton's equations for the integrable potentials V_λ.

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