DEFORMATION QUANTIZATION OF POLYNOMIAL POISSON ALGEBRAS

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Abstract. This paper discusses the notion of a deformation quantization for an arbitrary polynomial Poisson algebra $\mathbf{A}$. We compute an explicit third order deformation quantization of $\mathbf{A}$ and show that it comes from a quantized enveloping algebra. We show that this deformation extends to a fourth order deformation if and only if the quantized enveloping algebra gives a fourth order deformation; moreover we give an example where the deformation does not extend. A correction term to the third order quantization given by the enveloping algebra is computed, which precisely cancels the obstruction, so that the modified third order deformation extends to a fourth order one. The solution is generically unique, up to equivalence.

Contents

1. Introduction

Deformation theory for associative commutative algebras was first considered by Gerstenhaber in [10]. Its relevance for quantum mechanics was first pointed out in [1]; in the latter context one often speaks of deformation quantization. By definition a (formal) deformation of an associative commutative algebra $\mathbf{A}$ is an associative

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multiplication $\star$ on $\mathbf{A}^h = \mathbf{A}[[h]]$,

$$p \star q = pq + h\pi_1(p,q) + h^2\pi_2(p,q) + \cdots , \tag{1}$$

where pq denotes the original product of elements $p,\, q \in \mathbf{A}$. It is a fundamental fact that the associativity of $\star$ implies that the skew-symmetric part π_1^- of π_1 defines a Poisson bracket on $\mathbf{A}$, making it into a Poisson algebra: π_1^- is a biderivation, which means that

$$\pi_1^-(pq,r) = p\pi_1^-(q,r) + q\pi_1^-(p,r)$$

for any $p,q,r \in \mathbf{A}$, and it satisfies the Jacobi identity

$$\pi_1^-(\pi_1^-(p,q),r) + \pi_1^-(\pi_1^-(q,r),p) + \pi_1^-(\pi_1^-(r,p),q) = 0,$$

for any $p,q,r \in \mathbf{A}$. Since any deformation (1) is equivalent to a deformation for which π_1^- is skew-symmetric, the first fundamental question in this theory is whether every Poisson bracket on $\mathbf{A}$ appears in this way. Although it was already proven two decades ago that this is true when $\mathbf{A}$ is the Poisson algebra of functions on a symplectic manifold (see [5]), the general case was only settled recently (see [14]) as a consequence of the formality conjecture. In some sense, Kontsevich's proof in the case of Poisson structures on $C^\infty(\mathbb{R}^d)$ consists of a god-given formula, inspired by string theory, for which it is verified that it does indeed do the job.

The present paper is devoted on the one hand to a systematic construction of such a formula and on the other hand to an alternative approach, using enveloping algebras, to deformation theory. Our approach being algebraic, we will suppose that $\mathbf{A}$ is a polynomial algebra over a field $\mathbb{F}$ of characteristic zero, equipped with a Poisson structure. Since, as in most papers on this subject, all our computations are formal (i.e., the series $p \star q$ is not required to be convergent in h), the final formula also applies to $C^\infty(\mathbb{R}^d)$ as well.

Our computation of an explicit formula for π_2 is non-trivial in the sense that it involves the Jacobi identity. Already at this step the formula we obtain deviates from the one by Kontsevich: both formulas are equivalent in the sense that they define equivalent deformations, but a symmetric 2-cocycle has to be removed from Kontsevich's second order term in order to get our, simpler, formula. Using this simpler formula for π_2 we also arrive at a simple formula for π_3, as a result of a lot of non-trivial computations which not only involve the Jacobi identity but also its derivative. It is difficult to compare our third order term with Kontsevich's third order term, because the third order term depends very much on the choice of second order term and because Kontsevich's third order term has more than one hundred terms, with coefficients which are given by integrals that are hard to compute. Surprisingly enough our seemingly "natural" deformation does *not* (in general) extend to a fourth order deformation.

This fact is even more striking once one realizes that the third order deformation which we construct comes from a (quantized) universal enveloping algebra, making this deformation very natural. We define the enveloping algebra for any polynomial Poisson algebra $(\mathbf{A}, \{\cdot\,,\cdot\})$ as follows. First note that $\mathbf{A}$ can be seen as the symmetric algebra $\mathcal{S}(V)$ over a vector space V; then the Poisson bracket is a linear map $\mathcal{S}(V) \bigotimes \mathcal{S}(V) \to \mathcal{S}(V)$. We take the tensor algebra $\mathcal{T}(V)$ of V and we consider the two-sided ideal J^h of $\mathcal{T}(V)^h = \mathcal{T}(V)[[h]]$ generated by all elements of the form $x \otimes y - y \otimes x - h\sigma\{x,y\}$, where $x,y \in V$ and $\sigma : \mathcal{S}(V) \to \mathcal{T}(V)$ is the symmetrization map, defined by

$$\sigma\left(\prod_{i=1}^{n} a_i\right) = \frac{1}{n!}\sum_{p\in S_n} a_{p(1)}\otimes a_{p(2)}\otimes\cdots\otimes a_{p(n)}.$$

The quantized universal enveloping algebra is defined as $\mathcal{U}(V)^h = \mathcal{T}(V)^h/J^h$. In the case of a linear Poisson bracket we recover the usual definition of the enveloping algebra of a Lie algebra. It is a well-known but non-trivial fact that for a linear Poisson bracket the enveloping algebra does give a deformation quantization in the following way: the natural map $\mathcal{S}(V)^h \to \mathcal{U}(V)^h$ is a linear isomorphism, so the product on $\mathcal{U}(V)^h$ determines a product on $\mathcal{S}(V)^h$, which is a deformation quantization. In general, *i.e.*, for non-linear Poisson brackets, the map $\mathcal{S}(V)^h \to \mathcal{U}(V)^h$ fails to be injective, but surprisingly enough, for a general Poisson bracket it is injective precisely up to order three (in h). In fact, there is an obstruction to the injectivity of the map, which turns out to coincide with the obstruction which we found when trying to extend our third order deformation to a fourth order deformation. Thus the third order deformation which we construct using Hochschild cohomology extends to a fourth order deformation precisely when the quantized enveloping algebra gives a fourth order deformation. An explanation of this will be given in the text.

Since Kontsevich's third order deformation extends to a fourth order deformation while ours doesn't they cannot be equivalent. Indeed, we show how to add a biderivation to our third order term, without destroying associativity, so that the modified third order deformation can always be extended to a fourth order deformation. The check that it does depends on the Jacobi identity, and its first and second derivatives.

The structure of this paper is as follows. In Section 2 we fix the notation and we collect the properties that will be used. An explicit third order deformation for any polynomial Poisson algebra is computed and we find the obstruction to extending this deformation to a fourth order deformation. In the last paragraph of this section we show how to modify the third order deformation by a biderivation so that after modification it does extend to a fourth order deformation. In Section 3 we introduce the quantized enveloping algebra of a polynomial Poisson algebra, we prove a Poisson algebra version of the Poincaré-Birkhoff-Witt Theorem and we use it to show that our third order deformation, without the biderivation, comes from this enveloping algebra. In the final section a few examples with very different characteristics are worked out; in particular we give an example which shows that the third order deformation which is given by the quantized universal enveloping algebra does not extend in general.

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2. Construction of the universal deformation

2.1. **Preliminaries.** In this paper $\mathbf{A}$ will always denote a polynomial Poisson algebra (possibly in an infinite number of variables) over a field $\mathbb{F}$ of characteristic zero. $\mathbf{A}$ will often be viewed as the symmetric algebra over a vector space V. We fix a basis for V, which amounts to realizing $\mathbf{A}$ as $\mathbb{F}[x_j]_{j\in\mathcal{I}}$. For polynomials p and q we will denote their product pq by $\pi(p,q)$ and their Poisson bracket by $\{p,q\}$. Let h be a formal parameter and let $\mathbf{A}^h$ (resp. $\mathbb{F}[[h]]$) denote the algebra of formal power series with coefficients in $\mathbf{A}$ (resp. in $\mathbb{F}$). For $n\in\mathbb{N}$ we will also use the algebra $\mathbf{A}^h_n$ which is obtained from $\mathbf{A}^h$ by dividing out by the ideal generated by h^{n+1}. For elements $p,\, q\in\mathbf{A}^h$ we write $p=q \mod h^{n+1}$ when they project to the same element in $\mathbf{A}^h_n$.

Definition 2.1. An $\mathbb{F}[[h]]$-bilinear map $\pi_\star:\mathbf{A}^h\times\mathbf{A}^h\to\mathbf{A}^h$ is called a *(formal) deformation* of $\mathbf{A}$ when it satisfies the associativity condition

$$\pi_\star(\pi_\star(p,q),r)=\pi_\star(p,\pi_\star(q,r)) \tag{2}$$

for all p, q and r in $\mathbf{A}^h$ and reduces to π on $\mathbf{A}\cong\mathbf{A}^h_0$; *i.e.*, $\pi_\star(p,q)=\pi(p,q) \mod h$. More generally, when associativity merely holds on $\mathbf{A}^h_n$ we say that $\pi_\star$ defines an n-th order deformation.

When a (formal) deformation has the additional property that for any $p,q\in\mathbf{A}$ the product $q\star p$ is obtained from $p\star q$ by applying the involution of $\mathbf{A}^h$ determined by $h\mapsto -h$, then we say that it defines a (formal) *deformation quantization* of $\mathbf{A}$.

Two (n-th order or formal) deformations $\pi_\star$ and $\pi'_\star$ are called equivalent if there exists an $\mathbb{F}[[h]]$-linear map $F:\mathbf{A}^h\to\mathbf{A}^h$ such that $F(p)=p \mod h$ for any $p\in\mathbf{A}$ and such that $F(\pi_\star(p,q))=\pi'_\star(F(p),F(q))$ for any $p,\, q\in\mathbf{A}$.

If one writes $\pi_\star=\pi+h\pi_1+h^2\pi_2+\cdots$ then the associativity condition (2) can be expressed by an infinite list of relations

$$\delta\pi_n=\tfrac{1}{2}\sum_{i+j=n}[\pi_i,\pi_j],\qquad n=0,1,2,\dots, \tag{3}$$

where $[\cdot\,,\cdot]$ denotes the Gerstenhaber bracket (see [9]) and δ is the Hochschild coboundary operator (see [11]). Let us assume that the 2-cochains $\pi,\pi_1,\dots,\pi_{m-1}$ define an $(m-1)$-th order deformation of π; then (3) is satisfied for $n=1,\dots,m-1$ and the deformation extends to an m-th order deformation if and only if the m-th equation in (3) has a solution π_m. The latter condition is cohomological: it expresses that the 3-cocycle $\sum_{i+j=n}[\pi_i,\pi_j]$ is a coboundary; moreover, if a solution π_m exists it is unique up to addition of any 2-coboundary.

For polynomial algebras the condition for a 3-cocycle ψ to be a coboundary is that its skew-symmetrization ψ^- vanishes; moreover, if a 3-coboundary ψ is given by a tridifferential operator, then it is actually the coboundary of a bidifferential operator. Thus it suffices, in principle, to go inductively through the list (3), verifying at each step that the corresponding bidifferential operator π_i exists and then constructing such an π_i. One problem, however, is that the particular π_n chosen for the extension of the deformation to order n will have a pronounced impact on the further extendibility of the deformation. If the deformation does not extend to order $n+1$, it may be that a different choice of π_n would allow such an extension. Moreover, this effect is not limited to the next term in the extension, so that the extendibility of an extension up to order n is influenced by all of the choices of the cochains π_k for $1<k<n$.

For $i\in\mathcal{I}$, we denote by $\partial^i:\mathbf{A}\to\mathbf{A}$ the partial derivative with respect to x_i and we use $\partial^{i_1\cdots i_n}$ for repeated derivatives. They are used here to generate elements of $\mathrm{Hom}(\mathbf{A}^2,\mathbf{A})$ and $\mathrm{Hom}(\mathbf{A}^3,\mathbf{A})$, called differential 2-cochains and differential 3-cochains; the differential 2-cochains for example have the form $p_{i_1\dots i_m\dots i_n}\partial^{i_1\cdots i_m}\otimes$

$\partial^{i_{m+1}\cdots i_n}$, where we have used the summation convention and $p_{i_1\ldots i_m\ldots i_n} \in \mathbf{A}$ for any indices $i_1 \ldots i_n$. If ψ is a 2-cochain or a 3-cochain we denote its skew-symmetrization by ψ^-. For example $(\partial^i \otimes \partial^j)^- = \frac{1}{2}\left(\partial^i \otimes \partial^j - \partial^j \otimes \partial^i\right)$, which we also write as $\partial^i \wedge \partial^j$. Similarly $(\partial^i \otimes \partial^j \otimes \partial^k)^- = \partial^i \wedge \partial^j \wedge \partial^k$. For symmetric 2-cochains and 3-cochains we use a dot, e.g., $\partial^i \cdot \partial^j = \frac{1}{2}\left(\partial^i \otimes \partial^j + \partial^j \otimes \partial^i\right)$. In our computations below we use X_{ij} as a convenient notation for the Poisson bracket $\{x_i, x_j\}$ of the generators x_i and x_j of $\mathbf{A}$. The Poisson bracket can be written as $X_{ij}\partial^i \wedge \partial^j$, the Jacobi identity for π_1 reads

$$X^l_{ij}X_{kl} + X^l_{jk}X_{il} + X^l_{ki}X_{jl} = 0 \tag{4}$$

(for any i, j, $k \in \mathcal{I}$), the derivative of the Jacobi identity with respect to x_m is written as

$$X^{lm}_{ij}X_{kl} + X^{lm}_{jk}X_{il} + X^{lm}_{ki}X_{jl} + X^l_{ij}X^m_{kl} + X^l_{jk}X^m_{il} + X^l_{ki}X^m_{jl} = 0, \tag{5}$$

(for any i, j, k, $m \in \mathcal{I}$), and there are similar expressions for higher derivatives.

Finally we record explicit formulas for the Gerstenhaber bracket and for the Hochschild coboundary operator in terms of differential cochains. These formulas will be indispensable for our future computations. For a multi-index $I = (i_1, \cdots, i_n)$ of *order* $|I| = n$ and a polynomial $p \in \mathbf{A}$, denote $\partial^I = \partial^{i_1\cdots i_n}$ and $p^I = \partial^I(p)$. For a pair I, J of multi-indices, IJ will denote the multi-index resulting from concatenation. When summing over all multi-indices I and J such that $IJ = K$ it will be understood that there is only one term in the sum corresponding to the permutations of the elements of K that leave I and J invariant. If we denote

$$(p\partial^{I_1} \otimes \partial^{I_2})^J = \sum_{J_0 J_1 J_2 = J} p^{J_0}\partial^{I_1 J_1} \otimes \partial^{I_2 J_2}.$$

then the bracket of differential 2-cochains is given by

$$\begin{aligned} \left[p\partial^{I_1} \otimes \partial^{I_2}, q\partial^{J_1} \otimes \partial^{J_2}\right] = \\ p(q\partial^{J_1} \otimes \partial^{J_2})^{I_1} \otimes \partial^{I_2} - p\partial^{I_1} \otimes (q\partial^{J_1} \otimes \partial^{J_2})^{I_2} \\ + q(p\partial^{I_1} \otimes \partial^{I_2})^{J_1} \otimes \partial^{J_2} - q\partial^{J_1} \otimes (p\partial^{J_1} \otimes \partial^{I_2})^{J_2}. \end{aligned} \tag{6}$$

The Hochschild coboundary operator for a differential 2-cochains is given by

$$\delta(p\partial^J \otimes \partial^K) = p(\delta\partial^J) \otimes \partial^K - p\partial^J \otimes (\delta\partial^K), \tag{7}$$

where

$$\delta(p\partial^I) = -\sum_{I_1 I_2 = I} p\partial^{I_1} \otimes \partial^{I_2}.$$

For an n-differential operator $\delta^{I_1} \otimes \cdots \otimes \delta^{I_n}$, its *type* is the n-tuple $(|I_1|, \cdots, |I_n|)$.

2.2. **The third order deformation.** We will now show how a first order deformation $\pi + h\pi_1$, where $\pi_1 = \frac{1}{2}X_{ij}\partial^i \wedge \partial^j$ is a Poisson bracket on $\mathbf{A}$, can explicitly be extended to a third order deformation. Notice that $\pi + h\pi_1$ is indeed a first order deformation, a consequence of the fact that π_1 is a biderivation.

Proposition 2.2. *Given a first order deformation $\pi + h\pi_1$ of $\mathbf{A}$ where $\pi_1 = \frac{1}{2}X_{ij}\partial^i \wedge \partial^j$, let π_2 be the following symmetric cochain*

$$\pi_2 = \frac{1}{6}X^l_{ij}X_{lk}\partial^i \cdot \partial^{jk} + \frac{1}{8}X_{ij}X_{kl}\partial^{ik} \cdot \partial^{jl}. \tag{8}$$

Then $\pi + h\pi_1 + h^2\pi_2$ is a second order deformation of $\mathbf{A}$.

Proof. For n=2 the right hand side of equation (3) is $\frac{1}{2}[\pi_1,\pi_1]$. For any p, q, $r \in$ **A** one has $\frac{1}{2}[\pi_1,\pi_1]^-(p,q,r) = \frac{2}{3}(\pi_1(\pi_1(p,q),r)+\pi_1(\pi_1(q,r),p)+\pi_1(\pi_1(r,p),q))$, which is zero in view of the Jacobi identity. This shows that $\pi + h\pi_1$ extends to a second order deformation. Using equation (6) and the Jacobi identity (4) we find

$$\frac{1}{2}[\pi_1,\pi_1] = \frac{1}{4}X_{ik}^lX_{lj}\partial^i\otimes\partial^j\otimes\partial^k + \frac{1}{4}X_{ij}X_{kl}\left(\partial^{ik}\otimes\partial^l\otimes\partial^j - \partial^i\otimes\partial^k\otimes\partial^{jl}\right). \tag{9}$$

The third order part of $\delta\pi_2$ (with π_2 given by (8)) is computed using (7) to be given by

$$\frac{1}{12}X_{ij}^lX_{lk}\left(\partial^i\otimes\partial^j\otimes\partial^k + \partial^i\otimes\partial^k\otimes\partial^j - \partial^j\otimes\partial^k\otimes\partial^i - \partial^k\otimes\partial^j\otimes\partial^i\right).$$

Since i, j and k are just summation indices this can be rewritten as

$$\frac{1}{12}\left(X_{ij}^lX_{lk} + 2X_{ik}^lX_{lj} - X_{kj}^lX_{li}\right)\partial^i\otimes\partial^j\otimes\partial^k.$$

Using the Jacobi identity (4) this reduces to a single term

$$\frac{1}{4}X_{ik}^lX_{lj}\partial^i\otimes\partial^j\otimes\partial^k,$$

which is the third order term of $\frac{1}{2}[\pi_1,\pi_1]$. For the fourth order term one makes a similar computation (but the Jacobi identity is not used). □

One concludes from these computations that it is not obvious to guess a cochain whose coboundary is given; compare carefully (9) and (8).

Our next task is to find an explicit solution for the third equation in (3), namely the equation $\delta\pi_3 = [\pi_1,\pi_2]$; the existence of a solution follows from the fact that since π_1 is skew-symmetric and π_2 is symmetric, $[\pi_1,\pi_2]^- = 0$. The explicit computation of $[\pi_1,\pi_2]$ is long but straightforward. Writing the resulting tridifferential operator as a coboundary of a bidifferential operator is non-trivial and we will concentrate on this aspect. Clearly every term in $[\pi_1,\pi_2]$ is a tridifferential operator of (total) order 3, 4, 5 or 6. We will denote the i-th order part of a tridifferential operator by a subscript (i). We start with the highest order, which is the easiest.

Lemma 2.3. *The sixth order part of* $[\pi_1,\pi_2]$ *is the coboundary of a skew-symmetric 2-cochain,*

$$[\pi_1,\pi_2]_{(6)} = \delta\left(\frac{1}{48}X_{ij}X_{kl}X_{mn}\partial^{ikm}\wedge\partial^{jln}\right). \tag{10}$$

Proof. The sixth order terms in $[\pi_1,\pi_2]$ are the ones for which none of the coefficients in π_1 or π_2 are differentiated. There are twelve terms, they come from the bracket of π_1 and the fourth order term of π_2 only, and eight of them cancel in pairs, leaving the following expression for $[\pi_1,\pi_2]_{(6)}$.

$$\frac{1}{16}X_{mn}X_{ij}X_{kl}(\partial^{ikm}\otimes\partial^{jl}\otimes\partial^n + \partial^n\otimes\partial^{jl}\otimes\partial^{ikm} + \partial^{ikm}\otimes\partial^n\otimes\partial^{jl} + \partial^{jl}\otimes\partial^n\otimes\partial^{ikm}).$$

To compute $\delta(X_{ij}X_{kl}X_{mn}\partial^{ikm}\otimes\partial^{jln})$, use (7) and find twelve terms which come in equal triples due to the order three symmetry $(i,j)\to(k,l)\to(m,n)$. Formula (10) follows. □

Note that the computation did not involve the Jacobi identity. In the symplectic case this is the only term which survives. Next, we consider the terms of order 5.

Lemma 2.4. *The fifth order term* $[\pi_1,\pi_2]_{(5)}$ *is also the coboundary of a skew-symmetric 2-cochain, given by*

$$[\pi_1,\pi_2]_{(5)}=\frac{1}{12}\delta\left(X_{ij}^kX_{kl}X_{mn}\partial^{jlm}\wedge\partial^{in}\right).$$

Proof. The bracket $[\pi_1,\pi_2]_{(5)}$ has a lot of terms, they are of types (1,1,3), (1,3,1), (3,1,1), (1,2,2), (2,1,2) and (2,2,1). The terms of type $(1,3,1)$ cancel and in the other ones there is some simplification. By symmetry we only need to consider the terms of type (3,1,1), (1,2,2) and (2,1,2). We give the result below, omitting a global factor 1/24. Note the non-triviality of the coefficients.

$$\begin{aligned}
&(3,1,1):\left(X_{nm}X_{ij}^lX_{kl}+X_{im}X_{nj}^lX_{kl}\right)\partial^{jkm}\otimes\partial^i\otimes\partial^n,\\
&(1,2,2):\left(X_{ij}^kX_{kl}X_{mn}+2X_{ln}^kX_{km}X_{ij}+X_{nm}^kX_{kl}X_{ij}+3X_{ml}^kX_{kn}X_{ij}\right)\partial^n\otimes\partial^{jl}\otimes\partial^{mi},\\
&(2,1,2):\left(X_{ij}^kX_{kl}X_{mn}-X_{ij}^kX_{km}X_{ln}+3X_{ij}^kX_{kn}X_{ml}\right)\partial^{im}\otimes\partial^n\otimes\partial^{jl}.
\end{aligned}$$

It is surprising that all these terms integrate to a single term, *i.e.*, as a whole they can be written as

$$\delta\left(X_{ij}^kX_{kl}X_{mn}\left(\partial^{jlm}\otimes\partial^{in}-\partial^{in}\otimes\partial^{jlm}\right)\right). \tag{11}$$

Before checking this, note that (11) produces indeed precisely terms of the appropriate types. Clearly the (3,1,1) part of (11) is given by

$$X_{ij}^kX_{kl}X_{mn}\left(\partial^{jlm}\otimes\partial^i\otimes\partial^n+\partial^{jlm}\otimes\partial^n\otimes\partial^i\right)$$

and is easily rewritten in the form of the term of type (3,1,1). Type (2,1,2) involves the Jacobi identity. The (2,1,2) part of (11) is given by

$$\begin{aligned}
-X_{ij}^kX_{kl}X_{mn}(&\partial^{jl}\otimes\partial^m\otimes\partial^{in}+\partial^{jm}\otimes\partial^l\otimes\partial^{in}+\partial^{lm}\otimes\partial^j\otimes\partial^{in}\\
&+\partial^{in}\otimes\partial^j\otimes\partial^{lm}+\partial^{in}\otimes\partial^l\otimes\partial^{jm}+\partial^{in}\otimes\partial^m\otimes\partial^{jl}),
\end{aligned}$$

which is easily rewritten as

$$\begin{aligned}
(&X_{ij}^kX_{km}X_{nl}+X_{ij}^kX_{kl}X_{mn}+2X_{ij}^kX_{kn}X_{ml}\\
&+X_{nj}^kX_{ki}X_{ml}+X_{in}^kX_{kj}X_{ml})\partial^{im}\otimes\partial^n\otimes\partial^{jl}.
\end{aligned}$$

Now use the Jacobi identity (4) on the last two terms to obtain the term of type (2,1,2). Finally, the (1,2,2) part of (11) is given by

$$-X_{ij}^kX_{kl}X_{mn}\left(\partial^m\otimes\partial^{jl}\otimes\partial^{in}+\partial^j\otimes\partial^{lm}\otimes\partial^{in}+\partial^l\otimes\partial^{jm}\otimes\partial^{in}\right).$$

When this is rewritten as

$$\left(X_{ij}^kX_{kl}X_{mn}+X_{mn}^kX_{kl}X_{ij}+X_{ml}^kX_{kn}X_{ij}\right)\partial^n\otimes\partial^{jl}\otimes\partial^{mi}$$

then the first term matches with the first term of type (1,2,2) and the other two match up with the three remaining terms of type (1,2,2). ☐

For the fifth order term we used the Jacobi identity. For the fourth order term we will also use the derivative of the Jacobi identity (5).

Lemma 2.5. *The fourth order term* $[\pi_1,\pi_2]_{(4)}$ *is the coboundary of a skew-symmetric 2-cochain,*

$$[\pi_1,\pi_2]_{(4)}=\frac{1}{24}\delta\left(X_{lm}^kX_{jn}^lX_{ki}\partial^{mn}\wedge\partial^{ij}+X_{mn}^{kl}X_{lj}X_{ki}\partial^m\wedge\partial^{nij}\right).$$

Proof. As in the previous case we treat the terms in $[\pi_1,\pi_2]_{(4)}$ by type. There are just three types, to wit, (1,1,2), (1,2,1) and (2,1,1). By symmetry we only need to consider the terms of type (1,1,2) and (1,2,1). They have the following form (we omit the global constant 1/48).

$$(1,1,2): X_{ki}\left(4X^k_{ml}X^l_{jn}+2X^k_{lj}X^l_{mn}+2X^{kl}_{nj}X_{lm}+3X^{kl}_{mn}X_{lj}\right)\partial^m\otimes\partial^n\otimes\partial^{ij},$$
$$(1,2,1): 2\left(X^{kl}_{mn}X_{lj}X_{ki}-X^{kl}_{ij}X_{km}X_{ln}\right)\partial^m\otimes\partial^{ni}\otimes\partial^j.$$

We already simplified these formulas by using the Jacobi identity (for type (1,2,1) we used it twice). The verification for type (1,2,1) is now straightforward: the six terms of type (1,2,1) in

$$\delta\left(X^{kl}_{mn}X_{ki}X_{lj}\partial^m\otimes\partial^{nij}+X^{kl}_{ij}X_{km}X_{ln}\partial^{mni}\otimes\partial^j\right)$$

come in pairs and reduce to (1,2,1) above. The terms of type (1,1,2) in

$$\delta\left(2X^k_{lm}X^l_{jn}X_{ki}\partial^{mn}\otimes\partial^{ij}+X^{kl}_{mn}X_{ki}X_{lj}\partial^m\otimes\partial^{nij}\right)$$

are given by

$$X_{ki}\left(-2X^k_{ln}X^l_{jm}-2X^k_{lm}X^l_{jn}+X^{kl}_{mn}X_{lj}+2X^{kl}_{mj}X_{ln}\right)\partial^m\otimes\partial^n\otimes\partial^{ij}$$

which reduces to

$$X_{ki}\left(4X^k_{ml}X^l_{jn}+2X^k_{lj}X^l_{mn}+2X^{kl}_{nj}X_{lm}+3X^{kl}_{mn}X_{lj}\right)\partial^m\otimes\partial^n\otimes\partial^{ij}$$

by using the derivative of the Jacobi identity. □

Finally we consider the term of order 3. The proof does not involve the Jacobi identity and is left to the reader.

Lemma 2.6. *The third order term $[\pi_1,\pi_2]_{(3)}$ is also the coboundary of a skew-symmetric 2-cochain,*

$$[\pi_1,\pi_2]_{(3)}=\frac{1}{12}\delta(X_{ij}X^i_{kl}X^{jk}_{mn}\partial^n\wedge\partial^{lm}).$$

Our previous results lead to the following theorem.

Theorem 2.7. *Let $(\mathbf{A},\{\cdot,\cdot\})$ be a polynomial Poisson algebra with basis $\{x_i\}_{i\in\mathcal{I}}$ and denote $\pi_1=X_{ij}\partial^i\wedge\partial^j$, where $X_{ij}=\{x_i,x_j\}$. Then the following formula gives a third order deformation $\pi+h\pi_1+h^2\pi_2+h^3\pi_3$ of $\mathbf{A}$,*

$$\begin{aligned}\pi_\star=\pi&+\frac{h}{2}X_{ij}\partial^i\wedge\partial^j+\frac{h^2}{24}\left[4X^l_{ij}X_{lk}\partial^i\cdot\partial^{jk}+3X_{ij}X_{kl}\partial^{ik}\cdot\partial^{jl}\right]\\&+\frac{h^3}{48}[X_{ij}X_{kl}X_{mn}\partial^{ikm}\wedge\partial^{jln}+4X^k_{ij}X_{kl}X_{mn}\partial^{jlm}\wedge\partial^{in}\\&\quad+2X^{kl}_{mn}X_{lj}X_{ki}\partial^m\wedge\partial^{nij}+2X^k_{lm}X^l_{jn}X_{ki}\partial^{mn}\wedge\partial^{ij}\\&\quad+4X_{ij}X^i_{kl}X^{jk}_{mn}\partial^n\wedge\partial^{lm}].\end{aligned}\tag{12}$$

Up to equivalence every third order extension of $\pi+h\pi_1$ is of the form

$$\pi+h\pi_1+h^2(\pi_2+\varphi_2)+h^3(\pi_3+\varphi_3+\psi_3)\tag{13}$$

with φ_2 and φ_3 skew-symmetric biderivations and ψ_3 a symmetric 2-cochain satisfying $\partial\psi_3=[\pi_1,\varphi_2]$. Conversely, for such φ_2, φ_3 and ψ_3 (13) is always a third order deformation.

Proof. We proved already that (12) is a third order deformation. Suppose now that $\pi + h\pi_1 + h^2\pi_2' + h^3\pi_3'$ is another deformation which extends the same first order deformation. Then $\varphi_2 = \pi_2 - \pi_2'$ is a cocycle which can be assumed to be a skew-symmetric biderivation. Indeed, a symmetric 2-cocycle is always a coboundary and altering one term in a deformation by a coboundary leads to an equivalent deformation (of the same order). Since $\delta\pi_3' = [\pi_1, \pi_2 + \varphi_2]$ is a coboundary, $0 = [\pi_1, \pi_2 + \varphi_2]^- = [\pi_1, \varphi_2]^-$ and we can find a symmetric cochain whose coboundary is $[\pi_1, \varphi_2]$. Then $\pi_3' - \psi_3$ must differ from π_3 by a cocycle φ_3 which we may assume, again without loss of generality, to be a skew-symmetric biderivation. □

2.3. **The obstruction to a fourth order deformation.** In this section we want to investigate the fourth order term of the explicit deformation which is given by (12). For a given polynomial Poisson algebra $(\mathbf{A}, \{\cdot\,,\cdot\})$ we will denote the latter deformation by $\pi_\star = \pi + h\pi_1 + h^2\pi_2 + h^3\pi_3$; as before $\pi_1 = \frac{1}{2}\{\cdot\,,\cdot\} = \frac{1}{2}X_{ij}\partial^i \wedge \partial^j$.

Theorem 2.8. *The deformation (12) extends to a fourth (hence fifth) order deformation if and only if the following, non-trivial, condition is satisfied for any* $a < b < c \in \mathcal{I}$*:*

$$\begin{aligned}(14)\qquad & 2X_{ij}X^i_{kl}(X^{km}_{ab}X^{jl}_{cm} + X^{km}_{bc}X^{jl}_{am} + X^{km}_{ca}X^{jl}_{bm}) \\ & + X_{ij}X_{kl}(X^{ikm}_{ab}X^{jl}_{cm} + X^{ikm}_{bc}X^{jl}_{am} + X^{ikm}_{ca}X^{jl}_{bm}) = 0.\end{aligned}$$

Proof. The deformation (12) extends to a fourth order deformation if and only if $0 = [\pi_1, \pi_3]^- + \frac{1}{2}[\pi_2, \pi_2]^- = [\pi_1, \pi_3]^-$. The terms in $[\pi_1, \pi_3]^-$ have orders ranging from 3 to 8 only. We claim that the terms of order at least four all vanish, sketching the computation in the least trivial case when the order equals four. A direct application of (6) gives the following expression for the coefficient of $\partial^{a\bar a} \wedge \partial^b \wedge \partial^c$ in $[\pi_1, \pi_3]^-$ (some indices have been relabeled for later convenience and a global constant has been omitted; note also that a and $\bar a$ can be freely interchanged):

$$\begin{aligned}& 2X_{ji}X^i_{l\bar a}(X_{ak}X^{jkl}_{bc} + X_{bk}X^{jkl}_{ca} + X_{ck}X^{jkl}_{ab}) \\ & + 2X_{ji}X^i_{l\bar a}(X^{lj}_{ak}X^k_{bc} + X^{lj}_{bk}X^k_{ca} + X^{lj}_{ck}X^k_{ab}) \\ & + 2X_{jl}\left(X^{ij}_{\bar a b}(X_{ck}X^{kl}_{ia} + X_{ak}X^{kl}_{ci}) - X^{kl}_{ca}(X_{bi}X^{ij}_{k\bar a} + X_{\bar a i}X^{ij}_{bk})\right) \\ & + 2X^i_{k\bar a}X^j_{li}(X_{bj}X^{kl}_{ca} + X_{cj}X^{kl}_{ab}) - X^i_{k\bar a}X^{kl}_{bc}(X_{aj}X^j_{il} + X_{lj}X^j_{ia}) \\ & - 2X_{li}X^i_{jk}(X^j_{\bar a b}X^{kl}_{ca} - X^j_{ca}X^{kl}_{\bar a b}).\end{aligned}$$

We now use the second derivative of the Jacobi identity, i.e., we use the formula

$$(X_{ak}X^k_{cb} + X_{bk}X^k_{ac} + X_{ck}X^k_{ba})^{jl} = 0,$$

(valid for any indices a, b, c, j and l), to rewrite the first two lines (giving the first line below) and we twice use a derivative of the Jacobi identity to rewrite the third line (giving lines two and three below); the fourth line is simplified by a direct application of the Jacobi identity,

$$\begin{aligned}
&2(X_{ij}X^i_{l\bar a} + X_{il}X^i_{j\bar a})(X^j_{ak}X^{kl}_{bc} + X^j_{bk}X^{kl}_{ca} + X^j_{ck}X^{kl}_{ab})\\
&+ 2X_{lj}X^{ij}_{ab}(X^l_{ck}X^k_{i\bar a} + X^l_{\bar a k}X^k_{ci} + X^l_{ik}X^k_{\bar a c})\\
&- 2X_{lj}X^{kl}_{ca}(X^j_{bi}X^i_{k\bar a} + X^j_{\bar a i}X^i_{bk} + X^j_{ki}X^i_{\bar a b})\\
&+ 2X^i_{k\bar a}X^j_{li}(X_{aj}X^{kl}_{bc} + X_{bj}X^{kl}_{ca} + X_{cj}X^{kl}_{ab})\\
&- 2X_{li}X^i_{jk}(X^j_{\bar a b}X^{kl}_{ca} - X^j_{c\bar a}X^{kl}_{ab}).
\end{aligned}$$

Most terms in this expression cancel out in pairs, leaving

$$\begin{aligned}
&2(X_{jl}X^j_{ci} + X_{ji}X^j_{lc} + X_{jc}X^j_{il})X^i_{k\bar a}X^{kl}_{ab}\\
&\quad + 2(X_{jl}X^j_{ai} + X_{ja}X^j_{il})X^i_{k\bar a}X^{kl}_{bc}\\
&\quad + 2(X_{jl}X^j_{bi} + X_{ji}X^j_{lb} + X_{jb}X^j_{il})X^i_{k\bar a}X^{kl}_{ca}
\end{aligned}$$

which is zero, by a single application of the Jacobi identity on every line. It follows that the only non-zero terms in $[\pi_1,\pi_3]^-$ are terms of type $(1,1,1)$. Using (6) we find that the coefficient of $\partial^a\wedge\partial^b\wedge\partial^c$ in $[\pi_1,\pi_3]^-$ is given, (up to a constant) by the left hand side of (14); since this expression is skew-symmetric in a, b, c it will hold in general when it holds for $a<b<c\in\mathcal{I}$. Moreover, if (14) vanishes then π_4 can be chosen to be symmetric, which implies the existence of π_5, solution to $\delta\pi_5 = [\pi_1,\pi_4] + [\pi_2,\pi_3]$ because then $[\pi_1,\pi_4]^- = [\pi_2,\pi_3]^- = 0$. □

We will see later an example for which (14) is non-zero, showing that our deformation (12) in general does not extend to a fourth order deformation.

2.4. **The extension to a fourth order deformation.** We now show how the third order deformation that we have obtained needs to be modified in order to extend to a fourth order deformation. We denote the third order deformation quantization that we obtained in (12) by $\pi_\star = \pi + h\pi_1 + h^2\pi_2 + h^3\pi_3$ where $\pi_1 = \frac12\{\cdot\,,\cdot\}$. We have shown in Theorem 2.7 that we get, up to equivalence, all possible third order deformations of $(\mathbf{A},\{\cdot\,,\cdot\})$ by adding any biderivations φ_2 and φ_3 to π_2 and π_3 and adding any symmetric cochain ψ_3 satisfying $\delta\psi_3 = [\pi_1,\varphi_2])$ to π_3. Let us denote such an alternative deformation by $\pi'_\star = \pi + h\pi_1 + h^2\pi'_2 + h^3\pi'_3$. If $\pi'_\star$ extends to a fourth order deformation by adding a term $h^4\pi_4$ then π_4 is a solution to

$$\delta\pi_4 = [\pi_1,\pi'_3] + \frac12[\pi'_2,\pi'_2],$$

and the skew-symmetrization of the right hand side must vanish, leading to

$$[\pi_1,\pi_3]^- + [\pi_1,\varphi_3]^- + \tfrac12[\varphi_2,\varphi_2]^- = 0. \tag{15}$$

In view of the following lemma, all terms in the left hand side of (15) are of type (1,1,1).

Lemma 2.9. *If φ and ψ are two biderivations then $[\varphi,\psi]^-$ has type $(1,1,1)$.*

Proof. Let $\varphi = Y_{ij}\partial^i\wedge\partial^j$ and $\psi = Z_{kl}\partial^k\wedge\partial^l$. Then the piece of $[\varphi,\psi]$ that does not contain terms of type $(1,1,1)$ is given by

$$(Y_{ij}Z_{kl} + Y_{kl}Z_{ij})(\partial^{ik}\otimes\partial^l\otimes\partial^j - \partial^i\otimes\partial^k\otimes\partial^{lj}).$$

After skew-symmetrization every term appears twice with opposite signs hence they all cancel out. □

By computing the terms of type (1,1,1) in (15) we find that the existence of a fourth order deformation for a given $(\mathbf{A},\{\cdot,\cdot\})$ is equivalent to the existence of two skew-symmetric biderivations $\varphi_2=\frac{1}{4}Y_{ij}\partial^i\otimes\partial^j$ and $\varphi_3=\frac{1}{48}Z_{ij}\partial^i\otimes\partial^j$ such that for any $a<b<c\in\mathcal{I}$

$$X_{mc}Z^m_{ab}+Z_{mc}X^m_{ab}+6Y_{mc}Y^m_{ab}-X_{ij}X_{kl}X^{ikm}_{ab}X^{jl}_{cm}-2X_{ij}X^i_{kl}X^{km}_{ab}X^{jl}_{cm} + \text{ cycl } (a,b,c)=0. \tag{16}$$

Lemma 2.10. *The 2-cocycles $Y_{ab}=0$ and*

$$Z_{ab}=\frac{1}{2}X^{ik}_{ab}X^l_{ij}X^j_{kl}-X^{jk}_{ai}X^{il}_{bj}X_{kl},\quad (a,b\in\mathcal{I}) \tag{17}$$

solve equation (16) hence yield the correction term

$$\varphi_3=\frac{1}{96}(X^{ik}_{mn}X^l_{ij}X^j_{kl}-2X^{jk}_{mi}X^{il}_{nj}X_{kl})\partial^m\wedge\partial^n$$

to π_3 in (12) in order for the deformation quantization to extend to a fourth order deformation quantization.

Proof. Consider the following four equations, which are all a consequence of the Jacobi identity.

$$\begin{aligned}
1/2(X^i_{ab}X_{ci})^{jl}X^m_{jk}X^k_{lm} + \text{ cycl } (a,b,c)&=0,\\
(X^j_{ab}X_{jk}+X^j_{bk}X_{ja}+X^j_{ka}X_{jb})^{il}X^{km}_{ci}X_{lm} + \text{ cycl } (a,b,c)&=0,\\
(X^j_{ci}X_{jk}+X^j_{ik}X_{jc}+X^j_{kc}X_{ji})^lX^{im}_{ab}X^k_{lm} + \text{ cycl } (a,b,c)&=0,\\
(X^j_{ci}X_{jk}+X^j_{ik}X_{jc}+X^j_{kc}X_{ji})X^{im}_{al}X^{kl}_{bm} + \text{ cycl } (a,b,c)&=0.
\end{aligned}$$

Expand now $X_{mc}Z^m_{ab}+Z_{mc}X^m_{ab}+$ cycl (a,b,c), (where Z_{ab} is given by (17)) and add the above four equations. After the smoke clears up you will find

$$X_{ij}X_{kl}X^{ikm}_{ab}X^{jl}_{cm}+2X_{ij}X^i_{kl}X^{km}_{ab}X^{jl}_{cm} + \text{ cycl } (a,b,c)$$

as needed to solve (16). ∎

3. Deformation quantization via enveloping algebras

In this section we will show that the third order deformation which we constructed in Paragraph 2.2 for any polynomial Poisson algebra comes from a "quantized" enveloping algebra. The fact that an enveloping algebra appears here is not surprising. The symmetric algebra of a Lie algebra is a polynomial Poisson algebra in a natural way and it is well known that the quantized universal enveloping algebra of a Lie algebra is a deformation quantization of this Poisson algebra (see [1], [2]).

3.1. **The quantized universal enveloping algebra.** In order to describe the enveloping algebra of a polynomial Poisson algebra we will view polynomial algebras as symmetric algebras over a vector space. Let V be a (possibly infinite-dimensional) vector space over a field $\mathbb{F}$ of characteristic zero. For simplicity of notation we will denote elements in V by lowercase roman letters. For any positive integer n we let $V^n=V\otimes V\otimes\ldots\otimes V$ (n copies) and $V^0=\mathbb{F}$. The tensor algebra over V is the $\mathbb{Z}$-graded associative algebra (with unit) defined by

$$\mathcal{T}(V)=\bigoplus_{n=0}^{\infty}V^n.$$

The symmetric algebra $\mathcal{S}(V)$ is the quotient $\mathcal{S}(V) = \mathcal{T}(V)/I$, where I is the homogeneous ideal in $\mathcal{T}(V)$ generated by elements of the form $x \otimes y - y \otimes x$. The symmetric algebra is isomorphic to the polynomial algebra $\mathbb{F}[x_j]_{j\in\mathcal{I}}$ where $\{x_j\}_{j\in\mathcal{I}}$ is any basis for V. (Of course, any polynomial algebra can be represented in this form.) In particular, we will use juxtaposition to denote the product in $\mathcal{S}(V)$, just as we did for a polynomial algebra.

Any skew-symmetric map $V \otimes V \to \mathcal{S}(V)$ extends to a unique skew-symmetric biderivation on $\mathcal{S}(V)$. When this biderivation satisfies the Jacobi identity then $(\mathcal{S}(V), \{\cdot\,,\cdot\})$ becomes a polynomial Poisson algebra, and every polynomial Poisson algebra arises in this fashion. The quotient map $\mu : \mathcal{T}(V) \to \mathcal{S}(V)$ has a $\mathbb{F}$-linear right inverse $\sigma : \mathcal{S}(V) \to \mathcal{T}(V)$ which is defined by

$$\sigma\left(\prod_{i=1}^{n} a_i\right) = \frac{1}{n!}\sum_{p\in S_n} a_{p(1)} \otimes a_{p(2)} \otimes \cdots \otimes a_{p(n)},$$

where S_n is the symmetric group on n elements and $a_1, \ldots, a_n \in V$. We call σ the *symmetrization map.* Note that μ is an algebra homomorphism but the symmetrization map σ is not. Let $\mathcal{T}(V)^h$ ($\mathcal{S}(V)^h$) be the formal power series with coefficients in $\mathcal{T}(V)$ ($\mathcal{S}(V)$). Then $\mathcal{T}(V)^h$ and $\mathcal{S}(V)^h$ are naturally $\mathbb{F}[[h]]$-algebras, μ extends to an $\mathbb{F}[[h]]$-algebra homomorphism $\mu : \mathcal{T}(V)^h \to \mathcal{S}(V)^h$, and σ extends to a $\mathbb{F}[[h]]$-linear map $\sigma : \mathcal{S}(V)^h \to \mathcal{T}(V)^h$. Now we introduce a natural candidate for a deformation quantization of a polynomial Poisson algebra $(\mathcal{S}(V), \{\cdot\,,\cdot\})$.

Definition 3.1. Let J^h denote the two-sided ideal of $\mathcal{T}(V)^h$, generated by all elements

$$x \otimes y - y \otimes x - h\sigma\{x, y\} \qquad (x,\, y \in V). \tag{18}$$

The *quantized universal enveloping algebra* of $(\mathcal{S}(V), \{\cdot\,,\cdot\})$ is given by

$$\mathcal{U}(V)^h = \mathcal{T}(V)^h / J^h. \tag{19}$$

The induced product on $\mathcal{U}(V)^h$ is denoted by $\odot$ and the quotient map by

$$\rho : \mathcal{T}(V)^h \to \mathcal{U}(V)^h.$$

Thus, we have associated to a polynomial Poisson algebra $(\mathcal{S}(V), \{\cdot\,,\cdot\})$ a new (non-commutative) associative algebra $(\mathcal{U}(V)^h, \odot)$ and they are linked by the $\mathbb{F}[[h]]$-linear map (not a homomorphism!)

$$\tau : \mathcal{S}(V)^h \to \mathcal{U}(V)^h$$

given by $\tau = \rho \circ \sigma$. The maps τ, ρ and σ induce maps τ_n, ρ_n and σ_n on the quotient spaces $\mathcal{T}(V)^h_n$, $\mathcal{S}(V)^h_n$ and $\mathcal{U}(V)^h_n$ obtained by dividing out by the ideal (h^{n+1}). We also use the notation J^h_n for $J^h/(h^{n+1})$, so that $\mathcal{U}(V)^h_n = \mathcal{T}(V)^h_n / J^h_n$. We will see that in some important cases the map τ is a bijection, but that in general τ_n is only injective for $n \leq 3$. If τ is injective up to some order, the enveloping algebra provides a deformation quantization of $(\mathcal{S}(V), \{\cdot\,,\cdot\})$ of the same order, as given by the following theorem.

Theorem 3.2. *If $\tau : \mathcal{S}(V)^h \to \mathcal{U}(V)^h$ (resp. τ_n) is injective then the unique product $\star$ on $\mathcal{S}(V)^h$ which makes τ (resp. τ_n) into a homomorphism is a deformation quantization (resp. of order n) of the Poisson algebra $(\mathcal{S}(V), \{\cdot\,,\cdot\})$.*

Proof. τ is always surjective: simply note that $\mathcal{U}(V)^h_1$ is canonically isomorphic to $\mathcal{S}(V)$, so that for any $q \in \mathcal{U}(V)^h$ there exists a $p \in \mathcal{S}(V)$ such that $\tau(p) = q \mod h$. Then $\tau(p) - q = hq_1$, for some $q_1 \in \mathcal{U}(V)^h$. Continuing this process, we obtain a sequence of polynomials p_i such that $\tau(p + hp_1 + \cdots + h^k p_k) - q = h^k q_k$ for some $q_k \in \mathcal{U}(V)^h$. Then $\tau(p + hp_1 + \dots) = q$. It follows that τ_n is also surjective.

If τ_n is injective then the associative product which is induced by τ_n is given for $p, q \in \mathcal{S}(V)$ by

$$p \star q = \tau_n^{-1}(\tau_n(p) \odot \tau_n(q)).$$

We show that it defines a deformation of $(\mathcal{S}(V), \{\cdot\,,\cdot\})$ and that it is alternating. It is easy to see that

$$\tau_n(p) \odot \tau_n(q) = \tau_n(pq) \mod h$$

so that $p \star q = pq \mod h$; the associativity of $\star$ on $\mathcal{S}(V)^h_n$ implies that $p \star q = pq + h\pi_1(p,q) \mod h^2$ for some cocycle π_1. If we can show that π_1 is skew-symmetric then it is a biderivation and the fact that $\pi_1 = \frac{1}{2}\{\cdot\,,\cdot\}$ follows from the following check for elements $x, y \in V$,

$$h\pi_1(x,y) = \frac{1}{2}(x \star y - y \star x) = \frac{h}{2}\{x,y\} \mod h^2.$$

Now we show that $\star$ is alternating (up to order n), which proves in particular that π_1 is skew-symmetric. Let T be the anti-involution on $\mathcal{T}(V)^h$ induced by the map which reverses the order of elements in a tensor product, and let t be the involution of $\mathbb{F}[[h]]$ which is given by the map $h \mapsto -h$. Then t determines involutions of $\mathcal{S}(V)^h$ and $\mathcal{T}(V)^h$, which we will also denote by t. Let $\iota = T \circ t = t \circ T$, so ι is an anti-involution of $\mathcal{T}(V)^h$. Note that $T \circ \sigma = \sigma$. Thus $\iota(x \otimes y - y \otimes x - h\{x,y\}) = y \otimes x - x \otimes y - h\{y,x\}$, so ι maps the ideal J^h to itself inducing an anti-involution $\imath$. We also have the relations $\imath \circ \rho_n = \rho_n \circ \imath$ and $\tau_n \circ t = \imath \circ \tau_n$. Now $\star$ is alternating precisely when $t(p \star q) = q \star p$ for all p, q in $\mathcal{S}(V)$. But note that

$$\begin{aligned}\tau_n(t(p \star q)) &= \iota(\tau_n(p \star q)) = \iota(\tau_n(p) \odot \tau_n(q)) = \iota(\rho_n(\sigma_n(p)) \odot \rho_n(\sigma_n(q)))\\ &= \iota(\rho_n(\sigma_n(p) \otimes \sigma_n(q))) = \rho_n(\iota(\sigma_n(p) \otimes \sigma_n(q))) = \rho_n(\sigma_n(q) \otimes \sigma_n(p))\end{aligned}$$

and similarly $\tau_n(q \star p) = \rho_n(\sigma_n(q) \otimes \sigma_n(p))$. Since τ_n is an isomorphism, the conclusion follows. □

3.2. **The Poincaré-Birkhoff-Witt Theorem.** Theorem 3.2 demonstrates that the injectivity of τ_n is crucial. We show in the next theorem how injectivity of τ_n can be rephrased as an identity in $\mathcal{U}(V)^h_n$. Our proof is modeled on Birkhoff's proof of the Poincaré-Birkhoff-Witt Theorem (see [3]). Define a skew-symmetric map $\Delta : V^3 \to \mathcal{U}(V)^h$ by

$$\begin{aligned}\Delta(x,y,z) &= x \odot \tau\{y,z\} + y \odot \tau\{z,x\} + z \odot \tau\{x,y\}\\ &\quad - \tau\{y,z\} \odot x - \tau\{z,x\} \odot y - \tau\{x,y\} \odot z\end{aligned}$$

and call $\Delta = 0$ the *diamond relation.* For any n there is an induced map $\Delta_n : V^3 \to \mathcal{U}(V)^h_{n-1}$ and we call $\Delta_n = 0$ the *n-th diamond relation.* Note that for any $x, y, z \in V$,

$$hx \odot \tau\{y,z\} = x \odot y \odot z - x \odot z \odot y,$$

so that $h\Delta = 0$, and similarly $h\Delta_n = 0$ for all n. It is precisely the possibility of multiplying a nonzero element in $\mathcal{U}(V)^h$ by h to obtain zero that can cause τ

to fail to be injective, as we show in the theorem below. For the proof we need the notion of ordered elements in the tensor product. Fixing an ordered basis $\{x_i\}_{i\in\mathcal{I}}$ for V we call an element $\alpha = x_{i_1}\otimes x_{i_2}\otimes\cdots\otimes x_{i_m}\in\mathcal{T}(V)$ an *ordered* monomial if $i_1\le i_2\le\cdots\le i_m$, and strictly ordered if the inequalities above are strict inequalities. Let $\mathcal{O}(V)$ be the subspace of $\mathcal{T}(V)$ spanned by the ordered monomials, $\mathcal{O}(V)^h$ be the induced subspace of $\mathcal{T}(V)^h$, and $\mathcal{O}(V)^h_n=\mathcal{O}(V)^h/(h^{n+1})$ be the subspace of ordered elements in $\mathcal{T}(V)^h_n$. Also, for an element $\gamma\in\mathcal{T}(V)^h_n$, denote by $\gamma(0)$ its 0-th order part, so that $\gamma-\gamma(0)\in h\mathcal{T}(V)^h_n$.

Theorem 3.3. *For $n\ge 1$ the following four statements are equivalent.*

1. *τ_n is injective;*
2. *For any $\alpha\in\mathcal{U}(V)^h_n$, $h\alpha=0$ implies $\alpha=0 \mod h^n$;*
3. *$\star$ satisfies the n-th diamond relation $\Delta_n=0$;*
4. *The restriction of ρ_n to $\mathcal{O}(V)^h_n$ is injective.*

Moreover, each of these statements is true for $n=0$.

Proof. Let us first treat the case of $n=0$ because this is used later in the proof. The fact that τ_0 is injective follows immediately from the fact that the image of J^h in $\mathcal{T}(V)$ is the ideal I, so that τ_0 is essentially the identity map, from which it also follows that the restriction of ρ_0 to $\mathcal{O}(V)_0$ is injective. Statements 2) and 3) hold vacuously for $n=0$, so all statements are true for $n=0$.

Let us suppose that τ_n is injective and let $\alpha\in\mathcal{U}(V)^h_n$ be an element such that $h\alpha=0$. Since τ_n is surjective there exists $\beta\in\mathcal{S}(V)^h_n$ such that $\tau_n(\beta)=\alpha$. Then $\tau_n(h\beta)=h\tau_n(\beta)=0$, so that $h\beta=0$ and $\beta\in(h^n)$. Then $\alpha=\tau_n(\beta)=0 \mod h^n$, which shows that 1) implies 2).

That 2) implies 3) follows from the fact that $h\Delta_n=0$.

We now show that 4) implies 1), so we assume that the restriction of ρ_n to $\mathcal{O}(V)^h_n$ is injective. We show that τ_n is injective. By induction, we can assume that this theorem is true for $n-1$, so that τ_{n-1} is injective, since $\Delta_{n-1}=0$ if $\Delta_n=0$. Therefore, if $\tau_n(\gamma)=0$ for some $\gamma\in\mathcal{S}(V)^h_n$, then since $\tau_{n-1}(\gamma)=0$, we must have $\gamma=0 \mod h^n$. Thus $\gamma=h^np$ for some $p\in\mathcal{S}(V)$. But if $x_{i_1}\cdots x_{i_k}$ satisfies $i_1\le\cdots\le i_k$, then $\tau_n(h^nx_{i_1}\cdots x_{i_k})=h^n\rho_n(x_{i_1}\otimes\cdots\otimes x_{i_k})$, because we can always reorder the terms appearing in a tensor at the price of adding h times something. If we express $p=\sum_I a^Ix_{i_1}\cdots x_{i_k}$, where we sum over all increasing multi-indices $I=(i_1,\cdots,i_k)$, and $\beta=h^n\sum_I a^Ix_{i_1}\otimes\cdots\otimes x_{i_k}$, then $\beta\in\mathcal{O}(V)^h_n$ and satisfies $\rho_n(\beta)=\tau_n(\gamma)=0$, so that $\beta=0$, by injectivity of ρ_n on $\mathcal{O}(V)_n$. It follows that p must also vanish, and thus $\gamma=0$. This shows that 4) implies 1).

The rest of the proof is devoted to showing that 3) implies 4). We fix any $n\ge 1$ and assume that $\Delta_n=0$. Since the kernel of ρ_n restricted to $\mathcal{O}(V)^h_n$ is $\mathcal{O}(V)^h_n\cap J^h_n$, it suffices to show that $\mathcal{O}(V)^h_n\cap J^h_n\subseteq hJ^h_n$. An arbitrary element γ of $\ker\rho_n$ is of the form $\gamma=\gamma'+h\gamma''$ where $\gamma',\gamma''\in J^h_n$ and

$$\gamma'=\sum_{1\le l\le N}\alpha_l\otimes(x_{i_l}\otimes x_{j_l}-x_{j_l}\otimes x_{i_l}-h\sigma\{x_{i_l},x_{j_l}\})\otimes\beta_l \tag{20}$$

for some monomials α_l, β_l in $\mathcal{T}(V)$, basis elements x_{i_l}, and x_{j_l} and some positive integer N. We need to show that if γ is ordered then $\gamma'\in hJ^h_n$. We first show that $\gamma'(0)=0$. Since $\rho_n(\gamma)=0$ also $\rho_0(\gamma(0))=0$ which implies that $\gamma(0)=0$ because γ and hence also $\gamma(0)$ is ordered. Then $\gamma'(0)$ also vanishes because $\gamma(0)=\gamma'(0)$. Now consider a fixed multi-index I and define γ'_I by (20) but summing only over those l for which the indices in $\alpha_l\otimes x_{i_l}\otimes x_{j_l}\otimes\beta_l$ coincide with the ones in I (including multiplicities). Then evidently $\gamma_I'(0) = 0$. We will show that this implies that $\gamma_I' \in hJ_n^h$, from which it follows that $\gamma' \in hJ_n^h$ because $\gamma' = \sum_I \gamma_I'$.

First we consider the case when I is strictly ordered, in which case we may assume that $I = (1, \cdots, m)$ for some m. We denote by S_m the symmetric group and we consider its standard presentation with generators θ_k, $k = 1, \ldots, m-1$, (θ_k corresponds to the transposition $(k, k+1)$) and relations θ_k^2, $(\theta_l\theta_{l+1})^3$ and $(\theta_i\theta_j)^2$ for $|i-j| \geq 2$. For $\lambda \in S_m$, let $x_\lambda = x_{\lambda(1)} \otimes \cdots \otimes x_{\lambda(m)}$. Then we may express γ_I' as

$$\gamma_I' = \sum_{\lambda \in S_m} \sum_{k=0}^{m-1} a_{\lambda,k} \left(x_\lambda - x_{\theta_k\lambda} - h\chi_{\lambda,k}\right) \tag{21}$$

where $\alpha_{\lambda,k} \in \mathbb{F}$ and $\chi_{\lambda,k} = x_{\lambda(1)} \otimes \cdots \otimes \sigma\{x_{\lambda(k)}, x_{\lambda(k+1)}\} \otimes \cdots \otimes x_{\lambda(m)}$. Now consider the Cayley graph Γ_m of the above presentation for S_m. The vertices of Γ_m are given by the elements in S_m, with an edge connecting two vertices precisely when the permutations defining them differ by a transposition. The oriented edge connecting λ and $\theta_k\lambda$ is denoted by $e_{\lambda,k}$, so that $\partial(e_{\lambda,k}) = \lambda - \theta_k\lambda$. We define a linear map Ψ from the group $C^1(\Gamma_m, \mathbb{F})$ of (oriented) 1-chains on Γ_m to $\mathcal{T}(V)^h$ by letting

$$\Psi(e_{\lambda,k}) = x_\lambda - x_{\theta_k\lambda} - h\chi_{\lambda,k}.$$

Notice that Ψ is well-defined because although $e_{\theta_k\lambda,k}$ is the same edge as $e_{\lambda,k}$ but with the opposite orientation, it gets mapped to $-\Psi(e_{\lambda,k})$. Then obviously

$$\gamma_I' = \Psi\left(\sum_{\lambda \in S_m} \sum_{k=0}^{m-1} a_{\lambda,k} e_{\lambda,k}\right)$$

and the fact that $\gamma_I'(0)$ vanishes means that $\sum_{\lambda\in S_m}\sum_{k=1}^{m-1} a_{\lambda,k}e_{\lambda,k}$ is a cycle in the homology of the Cayley graph. By the universal coefficient theorem, every cycle (with coefficients in an arbitrary group) on a graph can expressed as a sum of multiples of closed edge paths in the graph; moreover, any cycle on the Cayley graph of a presentation is a sum of cycles (with integral coefficients) which correspond to the basic relations which appear in the presentation. It follows that $\sum_{\lambda\in S_m}\sum_{k=1}^{m-1} a_{\lambda,k}e_{\lambda,k} = \sum_{l=1}^{t} b_l r_l$ where each r_l corresponds to one of the basic relations appearing in the presentation and $\beta_l \in \mathbb{F}$. Therefore we have that

$$\gamma_I' = \sum_{l=1}^{t} b_l \Psi(r_l),$$

and it suffices to show that $\Psi(f) \in hJ^h$ for any cycle f which corresponds to a basic relation. First, notice that the cycle f which corresponds to θ_k^2 is zero because it consists of the sum of two copies of an edge with opposite orientation. Second, let i and j be such that $|i-j| > 1$ and let f_{ij} be the corresponding cycle, $f_{ij} = e_{\lambda,i} + e_{\theta_i\lambda,j} + e_{\theta_j\theta_i\lambda,i} + e_{\theta_j\lambda,j}$. Then

$$\Psi(f_{ij}) = -h(\chi_{\lambda,i} + \chi_{\theta_i\lambda,j} + \chi_{\theta_j\theta_i\lambda,i} + \chi_{\theta_j\lambda,j}).$$

Now both $\chi_{\lambda,i} + \chi_{\theta_j\theta_i\lambda,i}$ and $-\chi_{\theta_i\lambda,j} - \chi_{\theta_j\lambda,j}$ are given, up to an element of J_n^h, by

$$x_{\lambda(1)} \otimes \cdots \otimes \{x_{\lambda(i)}, x_{\lambda(i+1)}\} \otimes \cdots \otimes \{x_{\lambda(j)}, x_{\lambda(j+1)}\} \otimes \cdots \otimes x_{\lambda(m)},$$

showing that $\Psi(f_{ij}) \in hJ_n^h$. Finally, let us assume that f_l corresponds to the relation $(\theta_l\theta_{l+1})^3$. Then

$$f_l = e_{\lambda,l} + e_{\theta_l\lambda,l+1} + e_{\theta_{l+1}\theta_l\lambda,l} + e_{\theta_l\theta_{l+1}\theta_l\lambda,l+1} + e_{\theta_l\theta_{l+1}\lambda,l} + e_{\theta_{l+1}\lambda,l+1}$$

so that

$$\begin{aligned}\Psi(f_l) = hx_{\lambda(1)} \otimes \cdots \otimes (x_{\lambda(l)} \otimes \sigma\{x_{\lambda(l+1)}, x_{\lambda(l+2)}\} \\ - \sigma\{x_{\lambda(l+1)}, x_{\lambda(l+2)}\} \otimes x_{\lambda(l)} + \mathrm{cycl}) \otimes \cdots \otimes x_{\lambda(m)}.\end{aligned}$$

Since $\Delta_n = 0$ the term between parentheses lies in J_{n-1}^h. But now note that if $\alpha \in J_{n-1}^h$, then $\alpha = \beta + h^n\gamma$ for some $\beta \in J_n^h$, so that $h\alpha \in hJ_n^h$. Thus we can conclude that $\Psi(f_l) \in hJ_n^h$.

This completes the proof that 3) implies 4) in case I is strictly ordered. If $I = (i_1, \ldots, i_m)$ is merely ordered then the proof can repeated verbatim after replacing S_m with a quotient group, whose presentation is obtained from the above standard presentation of S_m by adding the relations θ_k for any k for which $i_k = i_{k+1}$. The corresponding Cayley graph is obtained from the one for S_m by collapsing the edges which correspond to those θ_k. □

The above theorem gives us an analytic criterion to check injectivity at some order. When we assume that injectivity at order $n-1$ has been checked then we may think of the n-th diamond relation as being a relation in $\mathcal{S}(V)_n^h$. Since this is the way in which we will use the diamond relation below, we formulate this fact in a separate theorem.

Theorem 3.4. *If $\tau_n : \mathcal{S}(V)_n^h \to \mathcal{U}(V)_n^h$ is injective (hence bijective) then τ_{n+1} is also injective if and only if the diamond relation*

$$x_a \star \{x_b, x_c\} - \{x_b, x_c\} \star x_a + \mathrm{cycl}(a,b,c) = 0$$

holds for any $a, b, c \in \mathcal{I}$. In this formula $\star$ is the product on $\mathcal{S}(V)_n^h$ which is induced using τ_n.

In this formulation the theorem will turn out to be very useful. For example we note that $p \star q = q \star p \mod h$ and conclude from it that τ_1 is injective.

3.3. **The $\star$-enveloping algebra.** In order to use the theorem to prove injectivity of the higher τ_i we need an explicit formula for the $\star$-bracket which comes from the enveloping algebra. We will show now that such a formula is given exactly by (12) and derive injectivity of τ_2 and τ_3 from it. For this purpose we associate an enveloping algebra $\mathcal{U}(V)_\star^h$ to a deformation $(\mathcal{S}(V)^h, \star)$ of $\mathcal{S}(V)$; in general $\mathcal{U}(V)_\star^h$ and $\mathcal{U}(V)^h$ will be different.

Definition 3.5. Let $(\mathcal{S}(V)^h, \star)$ be a deformation (of finite order or formal) of $\mathcal{S}(V)$ and denote the commutator in $(\mathcal{S}(V)^h, \star)$ by $[\cdot\,,\cdot]_\star$. Define $J_\star^h$ to be the two-sided ideal of $\mathcal{T}(V)^h$, generated by all elements of the form

$$a \otimes b - b \otimes a - \sigma[a,b]_\star, \qquad (a, b \in V)$$

and define the $\star$-*enveloping algebra* $\mathcal{U}(V)_\star^h$ of $(\mathcal{S}(V)^h, \star)$ by $\mathcal{U}(V)_\star^h = \mathcal{T}(V)^h / J_\star^h$.

For a given deformation $(\mathcal{S}(V)^h, \star)$ the enveloping algebras $\mathcal{U}(V)^h$ and $\mathcal{U}(V)_\star^h$ coincide if and only if

$$[x,y]_\star = h\{x,y\} \qquad (x, y \in V). \tag{22}$$

We call a deformation which satisfies (22) *bracket-exact.* In terms of the cocycles π_i this means that

$$\pi_i(x,y)=0 \qquad (x,y\in V,\, i>1).$$

For example, our general formula (12) defines a bracket-exact deformation quantization; adding any non-zero skew-symmetric biderivation to π_3 defines a deformation quantization which is not bracket-exact.

We now give a property which characterizes $\star$-enveloping algebras; in the case of bracket-exact deformations it characterizes enveloping algebras, showing that the $\star$-product which comes from the enveloping algebra is given by (12).

Definition 3.6. Let $(\mathcal{S}(V)^h,\star)$ be a deformation of $\mathcal{S}(V)$. The $\mathbb{F}[[h]]$-linear map,

$$\sigma_\star:\mathcal{S}(V)^h\to\mathcal{S}(V)^h$$

which is defined by

$$\sigma_\star\left(\prod_{i=1}^n a_i\right)=\frac{1}{n!}\sum_{p\in S(n)} a_{p(1)}\star a_{p(2)}\star\cdots\star a_{p(n)}.$$

is called $\star$-*symmetrization;* as in the definition of σ the elements $a_1,\dots,a_n$ belong to V. We will say that $\star$ is s-*balanced* if $\sigma_\star$ is the identity when restricted to elements of $\mathcal{S}(V)$ of degree $\leq s$. If $(\mathcal{S}(V)^h,\star)$ is a deformation (of order n) of $\mathcal{S}(V)$ then we call it a *balanced deformation* if $\star$ is s-balanced, where s is the degree of $[\cdot\,,\cdot]_\star$, i.e., the supremum of the degrees of all coefficients of $[x,y]_\star$, where x, y run over V (this degree may be infinite).

Note that when a deformation is bracket-exact then the degree of $[\cdot\,,\cdot]_\star$ is the degree of the corresponding Poisson bracket $\{\cdot\,,\cdot\}$.

Example 1. Any deformation is equivalent to a 2-balanced deformation. Indeed, such an equivalence is given precisely by $\sigma_\star$, i.e., define an equivalent product $\circ$ by

$$p\circ q=\sigma_\star^{-1}(\sigma_\star(p)\star\sigma_\star(q)).$$

Then

$$\sigma_\circ(xy)=\frac{1}{2}(x\circ y+y\circ x)=\frac{1}{2}\sigma_\star^{-1}(x\star y+y\star x)=xy,$$

for any x, $y\in V$, so that $\circ$ is 2-balanced.

Lemma 3.7. *Formula (12) gives, for any polynomial Poisson algebra, a bracket-exact balanced deformation of order 3.*

Proof. The proof of balancing is by induction. Obviously any deformation is 1-balanced, so we assume that the deformation, given by Formula (12), is n-balanced and prove that it is $(n+1)$-balanced. To do this, take a monomial a of degree $n+1$ and write $a=a_1a_2\cdots a_{n+1}$. We denote the associative product (12) on $\mathcal{S}(V)_3^h$ by $\star$ and the corresponding cochains by π_i. Using the associativity of $\star$ one has

$$\sum_{\tau\in S_{n+1}} a_{\tau(1)}\star a_{\tau(2)}\star\cdots\star a_{\tau(n+1)}=\sum_{i=1}^{n+1} a_i\star\left(\prod_{j\neq i}^{n+1} a_j\right)$$

so $\star$ is $(n+1)$-balanced when

$$\sum_{i=1}^{n+1} \pi_k \left(a_i, \prod_{j\neq i} a_j \right) = 0,$$

for $k = 1, 2, 3$. The verification is immediate. ☐

3.4. **Relating the deformations.** The following theorem gives a precise relation between balanced deformations and the $\star$-enveloping algebra.

Theorem 3.8. *If $(\mathcal{S}(V)^h, \star)$ is a balanced deformation of $\mathcal{S}(V)$ then the $\mathbb{F}[[h]]$-algebra homomorphism*

$$F : (\mathcal{T}(V)^h, \otimes) \to (\mathcal{S}(V)^h, \star)$$

which is induced by the natural inclusion $V \to \mathcal{S}(V)$ induces an $\mathbb{F}[[h]]$-algebra isomorphism

$$f : (\mathcal{U}(V)^h_\star, \odot) \to (\mathcal{S}(V)^h, \star).$$

When $(\mathcal{S}(V)^h, \star)$ is moreover bracket-exact then $\mathcal{U}(V)^h_\star = \mathcal{U}(V)^h$ and we have an isomorphism

$$f : (\mathcal{U}(V)^h, \odot) \to (\mathcal{S}(V)^h, \star).$$

The corresponding statements for n-th order deformations also hold.

Proof. We will only prove the first statement. If we denote the canonical map $\mathcal{T}(V)^h \to \mathcal{U}(V)^h_\star$ by $\rho_\star$ then it suffices to prove that $\ker F = \ker \rho_\star$ and that F is surjective. Let us first show that F is surjective. If $p \in \mathcal{S}(V)$ then there exists an element $\alpha \in \mathcal{T}(V)$ such that $p = F(\alpha) \mod h$. Indeed, since $\star$ is a deformation we have for any monomial $\prod_{i=1}^n a_i$ that

$$\prod_{i=1}^n a_i = a_1 \star a_2 \star \cdots \star a_n = F(a_1 \otimes a_2 \otimes \ldots \otimes a_n) \mod h.$$

More generally, for any $k \in \mathbb{N}$, since F is $\mathbb{F}[[h]]$-linear we can find $\alpha_0, \ldots, \alpha_k \in \mathcal{T}(V)$ such that $p = F(\alpha_0 + \alpha_1 h + \cdots + \alpha_k h^k) \mod h^{k+1}$. It follows that $\mathcal{S}(V) \subset \Im F$, which is sufficient to prove that F is surjective.

Let us show that $\ker \rho_\star = \ker F$. Take $a, b \in V$ and compute

$$\begin{aligned} F(a \otimes b - b \otimes a - \sigma[a,b]_\star) &= F(a) \star F(b) - F(b) \star F(a) - F\sigma[a,b]_\star \\ &= a \star b - b \star a - \sigma_\star[a,b]_\star \\ &= a \star b - b \star a - [a,b]_\star, \end{aligned}$$

which is zero; we used in the computation that $\sigma_\star = F\sigma$ and that $\sigma_\star[a,b]_\star = [a,b]_\star$ (because the deformation is balanced). This shows that $\ker \rho_\star \subset \ker F$.

To show that $\ker F \subset \ker \rho_\star$ we pick any $X \in \mathcal{T}(V)^h$ for which $F(X) = 0$ and show the existence of $Y \in \mathcal{T}(V)^h$ such that $\rho_\star(X) = \rho_\star(Y)$ and whose degree (in h) is larger than the degree of X. This will imply that for any $j \in \mathbb{N}$ the composition

$$\mathcal{T}(V)^h \xrightarrow{\rho_\star} \mathcal{U}(V)^h_\star \longrightarrow \mathcal{U}(V)^h_\star/(h^j)$$

maps X to 0, hence $\rho_\star(X) = 0$. To prove it, let d denote the degree of X, i.e., $X = X_0 h^d \mod h^{d+1}$. Let $\bar{X}_0$ denote the unique element in $\Im\sigma \cap \mathcal{T}(V)$ for which

$$\rho_\star(X_0) = \rho_\star(\bar{X}_0) \mod h$$

For simplicity of the notation, let us assume that $\bar{X}_0$ is of the form

$$\bar{X}_0 = \frac{1}{n!}\sum_{p\in S_n} a_{p(1)}\otimes a_{p(2)}\otimes\cdots\otimes a_{p(n)};$$

in general $\bar{X}_0$ will be a finite sum of such expressions. Then

$$\begin{aligned}F(\bar{X}_0) &= \frac{1}{n!}\sum_{p\in S_n} F(a_{p(1)})\star F(a_{p(2)})\star\cdots\star F(a_{p(n)})\\ &= \frac{1}{n!}\sum_{p\in S_n} a_{p(1)}\star a_{p(2)}\star\cdots\star a_{p(n)}\\ &= a_1a_2\dots a_n \mod h.\end{aligned}$$

Thus $F(X)=0$ implies that $\bar{X}_0=0$. So there exists a Y_0 such that $\rho_\star(X_0) = \rho_\star(hY_0) \mod h^2$ and hence there exists an element $Y\in\mathcal{T}(V)^h$ of the form $Y = Y_0h^{d+1} \mod h^{d+2}$ such that $\rho_\star(X)=\rho_\star(Y)$. □

We have seen that Formula (12) defines a bracket-exact balanced deformation (of order three). Theorem 3.8 implies that this deformation comes from the enveloping algebra, via the symmetrization map. This fact has the important consequence that we can use (12) to check injectivity of the maps τ_n. We already used the first term of our formula; *i.e.*, we have used $p\star q = pq \mod h$ to show that τ_1 is injective. Furthermore,

$$\begin{aligned}&\pi_1(x_a,\{x_b,x_c\})-\pi_1(\{x_b,x_c\},x_a)+\mathrm{cycl}(a,b,c)\\ &\qquad=\{x_a,\{x_b,x_c\}\}+\{x_b,\{x_c,x_a\}\}+\{x_c,\{x_a,x_b\}\}\end{aligned}$$

which is zero in view of the Jacobi identity. This proves injectivity of τ_2. Also

$$\pi_2(x_a,\{x_b,x_c\})-\pi_2(\{x_b,x_c\},x_a)+\mathrm{cycl}(a,b,c)=0$$

since π_2 is symmetric, hence τ_3 is also injective. The fact that this step is easy is similar to the fact that the existence of π_3 is automatic (given the fact that π_1 is skew-symmetric and that π_2 is symmetric). Finally, let us examine the injectivity of τ_4.

$$\begin{aligned}&\pi_3(x_a,\{x_b,x_c\})-\pi_3(\{x_b,x_c\},x_a)+\mathrm{cycl}(a,b,c)\\ &\quad=\frac{1}{24}(2X_{ij}X^i_{kl}X^{jk}_{ma}X^{lm}_{bc}+X^{kl}_{an}X_{lj}X_{ki}X^{nij}_{bc})+\mathrm{cycl}(a,b,c)\\ &\quad=\frac{1}{12}X_{ij}X^i_{kl}(X^{km}_{ab}X^{jl}_{cm}+X^{km}_{bc}X^{jl}_{am}+X^{km}_{ca}X^{jl}_{bm})\\ &\qquad\quad+\frac{1}{24}X_{ij}X_{kl}(X^{ikm}_{ab}X^{jl}_{cm}+X^{ikm}_{bc}X^{jl}_{am}+X^{ikm}_{ca}X^{jl}_{bm}).\end{aligned}$$

which is identical to the obstruction (14) which we found when trying to extend the deformation given by (12). We will see in the examples that in general the obstruction is non-zero, hence τ_4 is not injective and the enveloping algebra leads in general only to a deformation of order three.

4. Examples

In this section we will investigate some general and some more specific examples. We use the diamond relations to show that for constant and linear brackets the quantized enveloping algebra always gives a formal deformation quantization.

For the quadratic case we give a few examples in which the quantized enveloping algebra gives a fifth order deformation (at least) and we give an example in which the quantized enveloping algebra gives a formal deformation quantization. In the cubic case we give a few examples for which the quantized enveloping algebra gives a deformation of order three but not of higher order thereby showing the non-injectivity of τ_4 in general. All these examples are in $\mathbb{F}^4$ (with coordinates $x_1, \ldots, x_4$; $\mathbb{F}$ is a field of characteristic zero) but they have higher-dimensional counterparts. We will describe the Poisson structure by a 4×4 matrix whose (i,j)-th entry is the Poisson bracket $\{x_i, x_j\}$. We refer to this matrix as the *Poisson matrix*.

The simplest case is the one in which all X_{ij} are constant (i.e., they belong to $\mathbb{F}$). It is well-known that in this case a deformation quantization always exists. This follows also immediately from the diamond relations: since in this case

$$x \odot \tau\{y,z\} - \tau\{y,z\} \odot x = 0$$

for any $x, y, z \in V$ we conclude that $\Delta = 0$ hence that τ is injective. Alternatively it is immediate to check that the following explicit formula defines a deformation quantization in this case,

$$\pi_\star = \pi + \sum_{n=1}^{\infty} \frac{h^n}{2^n n!} X_{k_1 l_1} \cdots X_{k_n l_n} \partial^{k_1 \ldots k_n} \otimes \partial^{l_1 \ldots l_n}.$$

If a linear map $V \otimes V \to V$ satisfies the Jacobi identity then its extension to $\mathcal{S}(V)$ also satisfies the Jacobi identity, hence a Lie algebra leads in a natural way to a polynomial Poisson algebra. We call it *linear* because the bracket of any two basis elements is a linear combination of the basis elements. In this case it is known that the quantized enveloping algebra defines a formal deformation quantization. This is checked immediately using the diamond relations: in this case the fact that $\{y,z\} \in V$ for any $y, z \in V$ implies that

$$x \odot \tau\{y,z\} - \tau\{y,z\} \odot x = h\{x,\{y,z\}\} \tag{23}$$

so that the diamond relation holds in view of the Jacobi identity. Note also that, as a corollary of Theorem 3.8 all bracket-exact deformations of a linear bracket are isomorphic (to the one given by the enveloping algebra).

We can also consider brackets which have both linear and constant terms. Since the constant terms define a central extension of the linear terms this case is also covered by the linear case and the quantum enveloping algebra defines a deformation quantization. Alternatively, it is easy to see that (23) also holds in this case so that again the diamond relation is satisfied.

A major source of examples of non-linear polynomial Poisson brackets can be found on page 70 of [20]. Consider $\mathbb{C}^{2d}$ as the linear space of pairs of polynomials $(u(\lambda), v(\lambda))$ with $u(\lambda)$ monic of degree d and $v(\lambda)$ of degree less than d. If we write

$$\begin{aligned} u(\lambda) &= \lambda^d + u_1 \lambda^{d-1} + \cdots + u_{d-1}\lambda + u_d, \\ v(\lambda) &= v_1 \lambda^{d-1} + \cdots + v_{d-1}\lambda + v_d, \end{aligned}$$

then the following formula defines for any polynomial φ in two variables a Poisson bracket on $\mathbb{C}^{2d}$,

$$\begin{aligned} &\{u(\lambda), u_j\} = \{v(\lambda), v_j\} = 0, \\ &\{u_j, v(\lambda)\} = \varphi(\lambda, v(\lambda)) \left[\frac{u(\lambda)}{\lambda^{d-j+1}}\right]_+ \quad \mod u(\lambda), \qquad 1 \le j \le d. \end{aligned} \tag{24}$$

The subscript + means take the polynomial part and the expression $p(\lambda) \mod u(\lambda)$ means take the remainder obtained by Euclidean division. Since in these particular examples the Poisson matrix is always of the form $\begin{pmatrix} 0 & U \\ -U & 0 \end{pmatrix}$ we will only give the matrix U and the polynomial it derives from. Let us explain briefly how to compute U from (24) for a given bracket φ on $\mathbb{C}^4$. The coordinates are u_1, u_2, v_1 and v_2; also $u(\lambda) = \lambda^2 + u_1\lambda + u_2$ and $v(\lambda) = v_1\lambda + v_2$. Then the first row of U consists of the coefficients of $\varphi(\lambda, v(\lambda)) \mod u(\lambda)$ (just do Euclidean division) and the second row is given by the coefficients of $\varphi(\lambda, v(\lambda))(\lambda + u_1) \mod u(\lambda)$. For example, take $\varphi = x^3$. Then

$$U = \begin{pmatrix} u_1^2 - u_2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{pmatrix}.$$

In this case direct substitution in the left hand side of (14) gives zero so that the deformation, as given by (12), extends to a fifth order deformation. Another quadratic bracket is found by taking $\varphi = y$. Then U is given by

$$U = \begin{pmatrix} v_1 & v_2 \\ v_2 & u_1v_2 - u_2v_1 \end{pmatrix}.$$

Again (14) is satisfied. The same is also true for the sum, $\varphi = x^3 + y$, which corresponds to taking the sum of the above U matrices. Another quadratic example of interest is the quadratic bracket on $\mathfrak{gl}(2)$ (see [15]). It has Poisson matrix

$$\begin{pmatrix} 0 & x_1x_2 & 0 & x_2x_3 \\ -x_1x_2 & 0 & 0 & x_2x_4 \\ 0 & 0 & 0 & 0 \\ -x_2x_3 & -x_2x_4 & 0 & 0 \end{pmatrix}.$$

(14) is satisfied and the deformation extends to order five. In the following example of a quadratic bracket the quantized universal enveloping algebra gives a formal deformation quantization. If (a_{ij}) is a skew-symmetric matrix of size 4 then $\{x_i, x_j\} = a_{ij}x_ix_j$ defines a quadratic Poisson bracket on $\mathbb{C}^4$. In this case the relation

$$x_i \odot x_j - x_j \odot x_i = h\tau\{x_i, x_j\} = ha_{ij}(x_i \odot x_j + x_j \odot x_i)$$

can be rewritten as $x_j \odot x_i = A_{ij}x_i \odot x_j$ where $A_{ij} = (1 - ha_{ij})/(1 + ha_{ij})$. The verification of diamond relation then reduces to the following computation.

$$\begin{aligned}
&x_i \odot \{x_j, x_k\} - \{x_j, x_k\} \odot x_i + \mathrm{cycl}(i,j,k) \\
&\quad = x_i \odot x_j \odot x_k(a_{jk} - a_{ij}) + x_i \odot x_k \odot x_j(a_{jk} - a_{ki}) + x_j \odot x_i \odot x_k(a_{ki} - a_{ij}) \\
&\quad + x_j \odot x_k \odot x_i(a_{ki} - a_{jk}) + x_k \odot x_i \odot x_j(a_{ij} - a_{ki}) + x_k \odot x_j \odot x_i(a_{ij} - a_{jk}) \\
&\quad = x_i \odot x_j \odot x_k((a_{jk} - a_{ij}) + (a_{jk} - a_{ki})A_{jk} + (a_{ki} - a_{ij})A_{ij} \\
&\quad + (a_{ki} - a_{jk})A_{ik}A_{ij} + (a_{ij} - a_{ki})A_{ik}A_{jk} + (a_{ij} - a_{jk})A_{ij}A_{ik}A_{jk}) \\
&\quad = 0.
\end{aligned}$$

Therefore the quantized enveloping algebra of this quadratic Poisson bracket gives a formal deformation quantization.

Next we consider a few higher order brackets. As in the quadratic case, if you take $\varphi = x^4$ then

$$U = \begin{pmatrix} -u_1^3 + 2u_1u_2 & u_2^2 - u_1^2u_2 \\ u_2^2 - u_1^2u_2 & -u_1u_2^2 \end{pmatrix}.$$

In this case we find again that (14) is satisfied so that the enveloping algebra leads to a fifth order deformation. However, if you take $\varphi = y^2$ then U is given by

$$U = \begin{pmatrix} 2v_1v_2 - u_1v_1^2 & v_2^2 - u_2v_1^2 \\ v_2^2 - u_2v_1^2 & u_1v_2^2 - 2u_2v_1v_2 \end{pmatrix}.$$

and (14) is not satisfied: if we denote $x_1 = u_1$, $x_2 = u_2$, $x_3 = v_1$ and $x_4 = v_2$ then the left hand side of (14) is given by

$$-96x_3(x_4^4 - 2x_1x_3x_4^3 + 2x_2x_3^2x_4^2 - 2x_1x_2x_3^3x_4 + x_1^2x_3^2x_4^2 + x_2^2x_3^4)\,\partial^1 \wedge \partial^2 \wedge \partial^4.$$

It follows that in this case the quantized enveloping algebra only defines a third order deformation quantization. The choice $\varphi = y^2 + xy$ gives another non-zero term; basically any higher order polynomial leads to an obstruction. Also the cubic bracket on $\mathfrak{gl}(2)$ (see [15]), which is given by

$$U = \begin{pmatrix} 0 & x_1^2x_2 & x_2x_3^2 & x_2x_3(x_1+x_4) \\ -x_1^2x_2 & 0 & x_2x_3(x_4-x_1) & x_2x_4^2 \\ -x_2x_3^2 & x_2x_3(x_1-x_4) & 0 & x_2x_3^2 \\ -x_2x_3(x_1+x_4) & -x_2x_4^2 & -x_2x_3^2 & 0 \end{pmatrix}$$

leads to a non-zero obstruction, upon evaluating (14). Explicitly it is given by

$$96x_2^2x_3(2x_1x_4 + x_2x_3)(x_4 - x_1)$$
$$(x_3\partial^1 \wedge \partial^2 \wedge \partial^3 + (x_4 - x_1)\partial^1 \wedge \partial^2 \wedge \partial^4 - x_3\partial^2 \wedge \partial^3 \wedge \partial^4).$$

It follows that for most brackets the enveloping algebra only leads to a third order deformation.

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