1. Introduction

One of the oldest problems of Hamiltonian mechanics is to find the quadratures for integrable Hamiltonian systems. Kowalevski's calculations to find the quadratures for her top, as well as Kötter's enormous computations to get the quadratures for a rigid body in a fluid both required a lot of ingenuity and yet the geometry behind their calculations is totally obscure. Also their calculations seem special in each case and they do not provide insight in solving other examples by quadratures. In the last decades the use of algebraic geometry — especially the beautiful theory of Abelian varieties — in studying mechanical systems has revealed some of the mysteries and led to a better understanding of integrability and the role of geometry. In this article we discuss some of the aspects of the theory and we look at the following four questions:

1. Given two integrable systems, can one effectively find the birational transformation mapping one into the other (if any)?
2. Is there a systematic way to linearise an integrable system?
3. How can one calculate action-angle variables from the linearising variables?
4. How can one write the differential equations defining the integrable system in Lax-form?

These questions will be discussed in the context of two-dimensional *algebraic completely integrable systems* (*a.c.i. systems*). These are integrable systems for which the invariant (real) tori can be extended to complex algebraic tori (*Abelian surfaces*). This implies that algebraic geometry can be used to study these systems. For an extensive discussion of these systems we refer to [AvM1], a book which should be taken as a general reference on the subject and on the matter discussed below. In the second part of the text, we will give a list of examples to illustrate the techniques explained below to investigate the three questions. We shortly describe these examples now.

We will consider two problems of the standard form "kinetic energy T + potential energy V", where V_3 (*Hénon-Heiles potential*) and V_4 (*the quartic potential*) are polynomials of total degree 3 and 4 respectively. The Newton equations have the following form

$$V_3: \begin{cases} \ddot{q}_1 = 2q_1q_2, \\ \ddot{q}_2 = 2(2q_1^2 + 3q_2^2), \end{cases} \qquad V_4: \begin{cases} \ddot{q}_1 = \dfrac{q_1}{4}(2q_1^2 + 3q_2^2), \\ \ddot{q}_2 = q_2(3q_1^2 + 2q_2^2). \end{cases}$$

In appropriate coordinates the three body Toda lattice is governed by the equations

$$\begin{aligned} \dot{t}_1 &= t_1(t_5 - t_4), & \dot{t}_4 &= t_3 - t_1, \\ \dot{t}_2 &= t_2(t_6 - t_5), & \dot{t}_5 &= t_1 - t_2, \\ \dot{t}_3 &= t_3(t_4 - t_6), & \dot{t}_6 &= t_2 - t_3. \end{aligned}$$

In [BvM] the authors introduced the seven-dimensional system

$$\begin{aligned} \dot s_1 &= -8s_7, & \dot s_4 &= -4s_2s_5 - s_7,\\ \dot s_2 &= 4s_5, & \dot s_5 &= s_6 - 4s_2s_4,\\ \dot s_3 &= 2(s_4s_7 + s_5s_6), & \dot s_6 &= -s_1s_5 + 2s_2s_7,\end{aligned}$$
$$\dot s_7 = s_1s_4 + 2s_2s_6 - 4s_3,$$

to understand the geometry of an integrable top, the Goryachev-Chaplygin top. In appropriate variables Kowalevski's top is given by

$$\begin{aligned} \dot k_1 &= k_2k_3, & \dot l_1 &= 2k_3l_2 - k_2l_3,\\ \dot k_2 &= 2l_3 - k_1k_3, & \dot l_2 &= k_1l_3 - 2k_3l_1,\\ \dot k_3 &= -2l_2, & \dot l_3 &= k_2l_1 - k_1l_2.\end{aligned}$$

Finally g commuting vector fields on an arbitrary hyperelliptic Jacobian of genus g can be expressed as a Lax pair

$$\dot A = \frac{1}{2}[A, P_kA + B_k], \qquad A = \begin{pmatrix} v(x) & u(x) \\ w(x) & -v(x) \end{pmatrix}$$

where $k = 1, \ldots, g$ give the different flows. The entries of A are polynomials $u(x), v(x)$ and $w(x)$ of degrees $g, g-1$ and $g+2$, and $u(x)$ and $w(x)$ are monic. P_k is a linear operator on polynomials

$$P_k(\sum_{n=0}^{g+2} A_i x^i) = \sum_{i=0}^{k+1} A_{g-i+2} x^{1+k-i}$$

and the matrix B_k is a lower triangular matrix whose only non-zero entry is given by $-u_kx + 2u_1u_k - u_{k+1}$, where u_k is the coefficient of x^{g-k} in $u(x)$.

This Lax pair gives rise to another integrable system by taking the degrees of u, v, w as $g, g-1, g+1$ respectively and taking u and w monic; in this case, $b_k = -u_k$. We call these two integrable systems the *even master system* and the *odd master system* respectively. We called them *master systems* because indeed many two-dimensional systems can be reduced to them, as will be discussed later. The odd master system has been studied by Mumford when studying the KdV-equation (see [M]). The examples will be discussed in Section 6.

After a brief review of the fundamental tools in Section 2 we investigate the search for quadratures in a systematic way and the role of algebraic geometry in Section 3. For fixed generic values of the contants of motion one searches for Laurent solutions to the differential equations depending on a sufficient number of free parameters since these correspond to the points where the variables blow up. They thus correspond to the points on the divisor $\mathcal{D}$ which has to be adjoined to the affine invariant surface $\mathcal{A}$ defined by the constants of motion to get a complete (Abelian) variety $\mathcal{T}^2$, which in this paper we suppose to contain no elliptic curves. This divisor $\mathcal{D}$ can effectively be calculated by substituting these Laurent solutions in the invariants — in the two-dimensional case the irreducible components Γ_i

of this divisor are just curves. Again using the Laurent solutions one finds meromorphic functions on $\mathcal{T}^2$ having a certain pole at each of these curves Γ_i. A sufficient number of these functions will provide an embedding of the variety in projective space. From this the basic characteristics of the Abelian surface, like the polarisation induced by each of the curves on the surface, can be found.

(1) If one of the curves, say Γ, defines a principal polarisation on the surface, the polarised surface $(\mathcal{T}^2, \Gamma)$ is isomorphic to $(\mathrm{Jac}(\Gamma), t_x^*\Theta)$ for some translate $t_x^*\Theta$ of the theta divisor Θ (the zero locus of the classical Riemann theta function) on $\mathrm{Jac}(\Gamma)$. Using a non-trivial involution on the affine invariant surface $\mathcal{A}$ which fixes at least one point on Γ, one constructs an affine equation for the Kummer surface of $\mathcal{T}^2$. We show that by choosing coordinates in $\mathbb{P}^3$ such that $(0:0:0:1) \in \mathbb{P}^3$ corresponds to this fixed point, it is possible to construct explicitely the isomorphism from $(\mathcal{T}^2, \Gamma)$ to $(\mathrm{Jac}(\Gamma), t_x^*\Theta)$, where we use for $\mathrm{Jac}(\Gamma)$ the affine coordinates coming from natural coordinates on the symmetric product of Γ with itself. This map is called a *linearising map.* Finding this map is what is classically known as *solving the system by quadratures,* because in these symmetric coordinates, the differential equations reduce to

$$\begin{aligned} \frac{\dot{\mu}_1}{\sqrt{f(\mu_1)}} + \frac{\dot{\mu}_2}{\sqrt{f(\mu_2)}} &= a, \\ \frac{\mu_1\dot{\mu}_1}{\sqrt{f(\mu_1)}} + \frac{\mu_2\dot{\mu}_2}{\sqrt{f(\mu_2)}} &= b, \end{aligned} \tag{1}$$

for some constants $a, b \in \mathbb{C}$ and some equation $y^2 = f(x)$, $\deg(f) = 5$ or 6, for the curve. We call this representation the *Jacobi form of the differential equations.* As was shown in [M], from this representation the symmetric functions of μ_1 and μ_2, hence also the original variables defining the system, can be written down explicitely in terms of theta functions, themselves containing the roots of the polynomial defining the curve.

(2) Using the fact that every polarisation is induced by a principal polarisation via an isogeny (a homomorphism of Abelian surfaces with finite kernel), we are able to reduce the case of a general polarisation to the case of a principal polarisation, and we can proceed as in (1) to find the quadratures for the system.

The upshot is that we have a systematic way to linearise all two-dimensional a.c.i. systems. The method will be applied to the examples introduced above and we will show that the methods are also effective. For example, using these methods, we are able to linearsie Kowalevski's top in a very natural and systematic way. We also show in an example that for a system which is a.c.i. in the generalised sense, we can proceed in the same way. It is not clear however how the method generalises to higher-dimensional systems.

A second question (answered in Section 4) is the construction of action-angle variables for the real system underlying a two-dimensional a.c.i. system. This question is important, since action-angle variables lie at the base of the construction of the quantizised version of the system. Fixing one symplectic leaf, we are able to deduce from the Jacobi form of the differential equations conjugate variables ν_1 and ν_2 on the image of a torus neighborhood (in the real invariant manifold) of a given real torus in this leaf under the linearising map, such that the symplectic structure is the pull-back under the linearising map of the two-form

$$\omega = d\mu_1 \wedge d\nu_1 + d\mu_2 \wedge d\nu_2.$$

In the construction we make the natural assumption that the two functions which define the vector field of the system and a commuting vector field enter polynomially in some equation of the curve. Coordinates in which the symplectic structure takes the above canonical form are classically called *Darboux coordinates.* The functions ν_i are found by a simple integration and can be written down in terms of elementary functions in all examples we studied; the appearance of trancendental functions (such as logarithms) in the functions ν_i will also be clear from the discussion. Starting from these variables and using a standard method due to Arnold (see [A]), we can construct new Darboux coordinates p_1, p_2, ϕ_1, ϕ_2, having the property that the differential equations now take the extremely simple form

$$\begin{aligned} \dot{\phi}_i &= a_i, \\ \dot{p}_i &= 0, \end{aligned}$$

where the $a_i \in \mathbb{C}$ are constants. These variables are called *action-angle variables* since the ϕ_i are linear coordinates on the torus and the p_i have the dimension of action. Hence, we can construct explicit action-angle variables for these systems, involving (definite) integrals of the functions ν_i over intervals between Weierstrass points.

One of the observations, interesting from the point of view of differential geometry, is that often integrable systems have *compatible* Poisson brackets and moreover the basic vector field defining the system is Hamiltonian for these different brackets. Chapter 5will be devoted to give a necessary condition (which is easy to check) for the compatibility of Poisson brackets of a certain type, which we will find throughout the examples. Also we give a construction of compatible structures of this type, which applies for a large class of examples.

Finally, how can one find out how two systems are related? These mappings cannnot be found by mere inspection of the differential equations, but we will show how the linearisation leads to maps from a given system going with principally polarised Abelian surfaces, to either the even or the odd master system (thereby explaining the origin of the name), giving as a by-product an effective way to construct Lax equations for any two-dimensional integrable system (for which the generic invariant manifolds do not contain elliptic curves). More precisely we will find a map for every choice of two independent commuting Hamiltonians for the system. We also show how to relate the even and odd master system. The mappings are birational and quite complicated as will appear in the examples; also delicate covers may come into play and sometimes one only gets an isomorphism for restricted constants of motion (the moduli space of the invariant Abelian varieties may be smaller in one system than in the other). A Lax pair for the three body Toda lattice, different from the classical one, is given in this way. We obtain the following string of integrable systems:

$$\begin{array}{ccccc} & & \text{Ma} & & \\ & \nearrow & \updownarrow & \nwarrow & \\ \text{Toda} & \rightarrow & \text{7-dim} & \leftarrow & V_4 \end{array}$$

A second string, which is worked out in [AvM2] by totally different means contains integrable systems whose invariant Abelian surfaces carry a polarisation of type (1,2)

$$\begin{array}{ccccc} V_3 & & \rightarrow & & \text{Kow} \\ & \searrow & & \swarrow & \\ & & SO(4) & & \end{array}$$

($SO(4)$ stands for the geodesic flow on $SO(4)$ for the Manakov metric, see [H]). It would be worthwhile to compare their methods with the one discussed above. Also it would be interesting to look at polarisations of a different type, but for these only single examples are known for each polarisation type.

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2. Preliminaries

Let Γ be a smooth curve of genus g. We define two divisors D and D' in $\mathrm{Div}(\Gamma)$, the divisor group of Γ, to be *linearly equivalent,* $D \sim_l D'$, if and only if there exists a meromorphic function f on Γ for which $D - D' = (f)$. Here (f) stands for the divisor of zeroes minus the divisor of poles of f. It is well known that the *degree* of (f), $\deg(f)$, is zero, where $\deg\colon \mathrm{Div}(\Gamma) \to \mathbf{Z}$ is the homomorphism defined by $\deg(\sum_{i=1}^m n_i P_i) = \sum_{i=1}^m n_i$. Therefore

$$\mathrm{Pic}(\Gamma) \stackrel{\text{def}}{=} \frac{\mathrm{Div}(\Gamma)}{\sim_l} \cong \frac{\mathrm{Ker}\deg}{\sim_l} \oplus \mathbf{Z} \stackrel{\text{def}}{=} \mathrm{Jac}(\Gamma) \oplus \mathbf{Z}.$$

The groups $\mathrm{Pic}(\Gamma)$ and $\mathrm{Jac}(\Gamma)$ are called respectively the *Picard group* and the *Jacobian* of Γ. For any divisor D on Γ we denote by $\{D\}$ the equivalence class of all divisors on Γ which are linearly equivalent to D. For an effective divisor D we denote by $|D|$ the linear system of all effective divisors in $\{D\}$; we will also use this notation for effective divisors on surfaces, the notion of linear equivalence being defined in exactly the same way.

One of the most fundamental results in the theory of algebraic curves tells us that every Jacobi variety is a *principally polarised Abelian variety,* which we now explain. By an *Abelian variety* we mean a complex torus which can be embedded in projective space. To show that $\mathrm{Jac}(\Gamma)$ is an Abelian variety, we define a map $A\colon \otimes_s^g \Gamma \to \mathbb{C}^g/\Lambda$, the *Abel map* with respect to g independent holomorphic differentials $\vec{\omega} = {}^t(\omega_1, \ldots, \omega_g)$ on Γ, and arbitrary fixed points $Q_1, \ldots, Q_g$ on Γ by

$$A(\langle P_1, \ldots, P_g \rangle) = \sum_{i=1}^{g} \int_{Q_i}^{P_i} \vec{\omega} \bmod \Lambda, \tag{2}$$

where Λ is the lattice in $\mathbb{C}^g$ consisting of all vectors in $\mathbb{C}^g$ of the form $\oint_\gamma \vec{\omega}$, where γ runs over $H_1(\Gamma, \mathbf{Z})$. Since $\{\omega_1, \ldots, \omega_g, \bar{\omega}_1, \ldots, \bar{\omega}_g\}$ generate $H^{1,0} \oplus H^{0,1} = H^1_{DR}(\Gamma)$ (the first the Rham group of Γ), Λ is actually a lattice of maximal rank (called the *lattice of periods of* $\mathrm{Jac}(\Gamma)$) showing that $\mathbb{C}^g/\Lambda$ is a complex torus. By *Abel's theorem,* A is surjective and $\mathrm{Ker}A$ consists of those $\langle P_1, \ldots, P_g \rangle$ for which $P_1 + \cdots + P_g \sim_l Q_1 + \cdots + Q_g$. It follows that $\mathrm{Jac}(\Gamma)$ is a complex torus as well. To show that this torus can be embedded in projective space, one uses the *Kodaira embedding theorem,* which states that a compact complex manifold can be embedded in projective space if and only if it has a *Hodge form,* i.e., a closed, positive $(1,1)$-form whose cohomology class is rational. Applying this to a complex torus one obtains the famous *Riemann conditions*:

Theorem 1 *A complex torus $\mathbb{C}^g/\Lambda$ is an Abelian variety if and only if there exists an integral base $\{\lambda_1, \ldots, \lambda_{2g}\}$ for Λ and a complex base $\{e_1, \ldots, e_g\}$ for $\mathbb{C}^g$ such that $\Lambda = (\,\Delta_\delta\, Z\,)$, with $\Delta_\delta = \mathrm{diag}(\delta_1, \ldots, \delta_g)$ a diagonal matrix whose diagonal elements are positive integers satisfying $\delta_i \mid \delta_{i+1}$ and Z a symmetric matrix whose imaginary part $\Im(Z)$ is positive definite. In terms of coordinates $x_1, \ldots, x_{2g}$ dual to the base for Λ the Hodge form ω is given by*

$$\omega = \sum_{i=1}^{g} \delta_i dx_i \wedge dx_{i+g}.$$

Then choosing a unimodular base for $H_1(\Gamma)$ it follows from the classical reciprocity law for differentials of the first and third kinds that a base $\{\omega_1,\ldots,\omega_g\}$ for $H^0(\Gamma,\Omega^1\}$ can be chosen such that Λ takes the form $\Lambda = (\mathbf{1}\,Z)$ ($\mathbf{1} = \mathrm{diag}(1,\ldots,1)$). The first Riemann bilinear relation implies that Z is symmetric, and the second Riemann bilinear relation implies that $\Im(Z)$ is positive definite. By the Riemann conditions above, this shows that $\mathrm{Jac}(\Gamma)$ is an Abelian variety.

Secondly we explain what it means for $\mathrm{Jac}(\Gamma)$ to carry a principal polarisation. The cohomology class of a Hodge form ω on an Abelian variety is called a *polarisation.* The integers δ_i in $\omega = \sum \delta_i dx_i \wedge dx_{i+g}$ are invariants of the cohomology class of ω and are called the *elementary divisors* of the polarisation (different Hodge forms may give rise to different elementary divisors). By a slight abuse of language we sometimes say that the variety has *type* $(\delta_1,\ldots,\delta_g)$ meaning that the variety has a Hodge form with elementary divisors $(\delta_1,\ldots,\delta_g)$. With this convention, an Abelian variety is said to carry a *principal polarisation* when it has type $(1,\ldots,1)$ and we see that $\mathrm{Jac}(\Gamma)$ carries a principal polarisation. It can be shown that every principally polarised Abelian surface which does not contain elliptic curves is isomorphic to the Jacobian of a curve of genus two (see [GH]).

The embedding of an Abelian variety $\mathcal{T}^g$ can be made concrete by using line bundles. Given any Hodge form on $\mathcal{T}^g$, there exists a (positive) line bundle L on $\mathcal{T}^g$ whose first Chern class $c_1(L)$ is exactly the cohomology class of ω and conversely the first Chern class of any positive line bundle gives a polarisation on $\mathcal{T}^g$. As an application it is easy to check that the elementary divisors $(\delta'_1,\ldots,\delta'_g)$ corresponding to the k-fold power $L^{\otimes k}$ of a line bundle L with elementary divisors $(\delta_1,\ldots,\delta_g)$ of L are related by $(\delta'_1,\ldots,\delta'_g) = (k\delta_1,\ldots,k\delta_g)$. If L is a positive line bundle on $\mathcal{T}^g$ and $(\delta_1,\ldots,\delta_g)$ are the elementary divisors of the polarisation $c_1(L)$ then

$$\dim H^0(\mathcal{T}^g,\mathcal{O}(L)) = \det \Delta_\delta = \delta_1\delta_2\cdots\delta_g \stackrel{\text{def}}{=} \delta \tag{3}$$

Associated to L is a map from $\mathcal{T}^g$ to $\mathbb{P}^{\delta-1} \cong \mathbb{P}H^0(\mathcal{T}^g,\mathcal{O}(L))$ defined as follows. Fixing a base $\{s_1,\ldots,s_\delta\}$ for $H^0(\mathcal{T}^g,\mathcal{O}(L))$, the point $(\phi^*(s_1)(P):\cdots:\phi^*(s_\delta)(P))$ is independent from the chosen trivialisation ϕ for L around a point $P\in\mathcal{T}^g$, thereby defining a holomorphic mapping as long as in each of the points at least one of the sections s_i does not vanish, i.e., the complete linear system $H^0(\mathcal{T}^g,\mathcal{O}(L))$ has no base points. By a result due to Lefschetz, $L^{\otimes 3}$ embeds for any positive line bundle L over $\mathcal{T}^g$. Under the basic correspondence between line bundles and divisors, the space $H^0(\mathcal{T}^g,\mathcal{O}(L))$ is identified with the space $\mathcal{L}(\mathcal{D}) = \{f\in\mathcal{M}(\Gamma) | (f) \geq -\mathcal{D}\}$. Since for our purposes it is most natural to work with $\mathcal{L}(\mathcal{D})$, we will always embed our Abelian varieties in $\mathbb{P}\mathcal{L}(\mathcal{D})$. When we take for example the polarisation induced by the line bundle which corresponds to an embedding of a genus two curve Γ into its Jacobian, then we can embed $\mathrm{Jac}(\Gamma)$ in $\mathbb{P}^8$ by the 9 functions with a 3-fold pole along the embedded curve at worst. The image of the holomorphic map defined by the functions in $\mathcal{L}(2\Gamma)$ is for a generic Jacobian a quartic surface isomorphic to the *Kummer surface* of $\mathrm{Jac}(\Gamma)$, which is defined for a general Abelian variety $\mathcal{T}^g$ as the quotient surface $\mathcal{T}^g/\jmath$, where $\jmath$ is the (-1)-involution given by $(z_1,\ldots,z_g)\mapsto(-z_1,\ldots,-z_g)$ in the natural coordinates coming from the universal covering space $\mathbb{C}^g$ of $\mathcal{T}^g$. Clearly the Kummer surface has sixteen singular points which correspond to the two-torsion points of $\mathcal{T}^2$.

The main interest in this paper is in the two-dimensional case. The situation here is very special since every Abelian surface is an unramified cover of a hyperelliptic Jacobian: at the one hand every Abelian variety is obviously an unramified cover of a principally polarised Abelian variety, which is the Jacobian of a curve (of genus two) in case the dimension of the Abelian variety is two, and at the other hand every curve of genus two is hyperelliptic; in suitable coordinates an equation for a hyperelliptic curve of genus two is given by $y^2 = f(x)$, where $\deg f = 5$ or 6. In these coordinates a base for the space of holomorphic differentials is given by $\{\frac{dx}{y}, \frac{xdx}{y}\}$ and the hyperelliptic involution is given by $(x, y) \mapsto (x, -y)$ and we see that the curve has six fixed points the *Weierstrass points of* Γ, one of which is the point at infinity when $\deg f = 5$. The following two lemmas give a precise and explicit description of $\mathrm{Jac}(\Gamma)$ in terms of simple divisors on Γ. For a proof we refer to [M].

Lemma 2. *Let Γ be a smooth curve of genus two and denote by σ the hyperelliptic involution on Γ. Then for any two different divisors $P_1 + P_2$ and $Q_1 + Q_2$ on Γ, $P_1 + P_2 \sim_l Q_1 + Q_2$ if and only if $P_1 = \sigma P_2$ and $Q_1 = \sigma Q_2$.*

Lemma 3. *Fixing any two points Q_1 and Q_2 on Γ, every point on $\mathrm{Jac}(\Gamma)$ is of the form $\{P_1 + P_2 - Q_1 - Q_2\}$ for some points P_1 and P_2 on Γ. This representation is unique if and only if $P_1 \neq \sigma P_2$, i.e., all divisors of the form $P + \sigma P - Q_1 - Q_2$ $(P \in \Gamma)$ are linearly equivalent. Therefore $\mathrm{Jac}(\Gamma)$ is obtained from $\Gamma \otimes_s \Gamma$ by blowing down the fundamental pencil $\{\langle P, \imath P\rangle \mid P \in \Gamma\}$ using the Abel map A.*

To study general Abelian surfaces using curves on these surfaces and sections of their line bundles we recall (for example from [GH]) that if $\mathcal{D}$ is an effective divisor on an Abelian surface $\mathcal{T}^2$, and if $L = [\mathcal{D}]$ then

$$\delta_1\delta_2 = \dim H^0(\mathcal{T}^2, \mathcal{O}(L)) = \dim \mathcal{L}(\mathcal{D}) = \frac{\mathcal{D}.\mathcal{D}}{2} = g(\mathcal{D}) - 1, \tag{4}$$

where $g(D)$ denotes the virtual genus of D (which equals the topological genus of D if D is non-singular). A final remark which will be useful later is that any genus two curve on $\mathrm{Jac}(\Gamma)$ is a translate of the Riemann theta divisor, in particular the embedded curve is isomorphic to Γ.

3. The linearisation procedure

In this section we present our systematic method to linearise a two-dimensional a.c.i. system. Since this amounts to the construction of special coordinates, we will first construct a set of coordinates on some affine part $\mathcal{A}$ of a two-dimensional Jacobian $\mathrm{Jac}(\Gamma)$, obtained by removing two touching translates of the theta divisor, thereby generalising Mumford's construction which concerns the case in which both translates coincide (Proposition 6). We show moreover that all sets of coordinates on this affine space are in fact constructed in that way (whether or not the two translates are coincident or not) and we prove a geometrical result (Proposition 7) which leads to a construction of these coordinates in terms of the variables in which there are given equations defining the affine part $\mathcal{A}$. These coordinates are closely related to symmetric function on (some affine part of) the curve Γ and we prove (Theorem 9) that any holomorphic vector field linearises in these variables. This is applied to linearise a two-dimensional a.c.i. system as follows.

For fixed generic constants of the motion, the structure of the divisor to be adjoined at infinity to complete the affine invariant torus defined by the invariant polynomials of the dynamical system can easily be deduced from the Laurent solutions to the differential equations, as will be illustrated in the examples. This divisor at infinity consists of different curves, among which one finds in most (two-dimensional) systems one single or two touching curves of genus two or unramified covers of these; in any case[†] the divisor at infinity is always linearly equivalent to such a divisor so that by changing the divisor at infinity we are in the former situation again (see the Kowalevski top). If one curve or two curves of genus two are found, $\mathcal{A}$ is the affine part of a Jacobian and we may by the above method relate the variables defining the system to linearising variables, since the invariants of the system give equations for an affine part $\mathcal{A}$ as above, thereby linearising the system. The case in which covers of curves of genus two are found relates to more general Abelian surfaces and is discussed at the end of this section.

We start with a proposition which controls the position of the curve in its Jacobian. We need the following lemma:

Lemma 4. *Fixing any point R on Γ, define two embeddings $\imath_1$ and $\imath_2$ of Γ into $\mathrm{Jac}(\Gamma)$ by $\imath_1(P)=\{P-R\}$ and $\imath_2(P)=\{P-\sigma R\}$. Then $\imath_1(\Gamma)$ and $\imath_2(\Gamma)$ either coincide or are tangent at their only intersection point $O=\{0\}$. If τ_v is any translation (by $v\in\mathrm{Jac}(\Gamma)$) then the same holds for $\tau_v\imath_1(\Gamma)$ and $\tau_v\imath_2(\Gamma)$, the point O being replaced by v.*

Proof

Using Lemma 2 it is easy to show that $\imath_1$ and $\imath_2$ define embeddings of the curve in its Jacobian. If $R=\sigma R$, i.e., R is a Weierstrass point, then trivially both curves coincide. Otherwise the curves intersect in the points $\{P-R\}$ for which there is a $Q\in\Gamma$ such that $\{P-R\}=\{Q-\sigma R\}$, i.e., $P+\sigma R\sim_l Q+R$. Then it follows from Lemma 2 that

[†] Recall that throughout this paper we suppose that a generic member of the family of Abelian surfaces we consider does not contain elliptic curves, i.e., is not isogeneous to a product of elliptic curves.

$P = \sigma^2 R = R$ and $Q = \sigma R$ giving the origin O as the unique intersection point. Note however that the curves $\imath_1(\Gamma)$ and $\imath_2(\Gamma)$ intersect in two points since

$$\frac{\imath_1(\Gamma).\imath_2(\Gamma)}{2} = g(\imath_1(\Gamma)) - 1 = 1,$$

showing that the curves are actually tangent at O. ∎

Proposition 5 *Let R be any fixed point on Γ and let $\imath_1$ be the corresponding embedding of Γ in its Jacobian as in the above lemma, then the translation which sends the curve $\imath_1(\Gamma)$ to Γ maps the origin to R.*

Proof

Let τ_v denote the unique translation which maps O to $R \in \Gamma$ and let τ_{-v} denote the inverse translation. Let $w \in \mathrm{Jac}(\Gamma)$ be such that $w + \tau_{-v}\Gamma = \imath_1(\Gamma)$, then it follows from $O \in \tau_{-v}\Gamma$ that $w = \{S - R\}$ for some $S \in \Gamma$. Therefore $\tau_{-v}\Gamma = \imath_1(\Gamma) - \{S - R\}$ consists of the points $\{P - R\} - \{S - R\} = \{P - S\}$ where P runs through Γ. Since in the equality $\Gamma = \tau_v\{\{P - S\} \mid P \in \Gamma\}$ the point R corresponds to $\tau_v\{0\}$ it follows that $\{R - S\} = 0$, i.e., $R = S$. ∎

In the sequel we fix the curve Γ and the point R on it. The unique curve tangent to Γ at v (which is Γ itself when R is a Weierstrass point) will be denoted by Γ_2 and for symmetry in the notation, Γ will often be denoted by Γ_1, the notation Γ being reserved for the cases in which the particular embedding is irrelevant. Also we fix any holomorphic cover $\pi : \Gamma \to \mathbb{P}^1$ of order two which maps R (hence also σR) to $\infty \in \mathbb{P}^1$, and denote the points R and σR by the more convenient notations ∞_1 and ∞_2. Remark that in the new notations ∞_1 is a Weierstrass point iff $\infty_1 = \infty_2$ iff $\Gamma_1 = \Gamma_2$.

We first define the bijection between an affine part $\mathcal{A}$ of $\mathrm{Jac}(\Gamma)$ and a smooth space of pairs of polynomials, thereby constructing coordinates on $\mathcal{A}$. It generalises Mumford's construction, which consists in the case $\infty_1 = \infty_2$, $v = 0$.

Proposition 6 *Let Q_1 and Q_2 be points on the curve Γ for which $v = \{\infty_1 + \infty_2 - Q_1 - Q_2\}$. Then the map*

$$\begin{aligned}
\mathcal{A} &= \mathrm{Jac}(\Gamma) \setminus (\Gamma_1 + \Gamma_2) \\
&= \{P_1 + P_2 - Q_1 - Q_2 \mid \\
&\qquad P_i, Q_i \in \Gamma, \\
&\qquad i \neq j \Rightarrow P_i \neq \sigma P_j, \\
&\qquad P_i \neq \infty_1, \infty_2\}
\end{aligned}
\quad \longrightarrow \quad
\begin{aligned}
&\{(u,v) \mid u(x) = x^2 + u_1 x + u_2, \\
&\qquad v(x) = v_1 x + v_2, \\
&\qquad u(x) \mid f(x) - v^2(x)\}, \\
&u(x) = (x - x(P_1))(x - x(P_2)), \\
&v(x) = \frac{(x(P_1) - x)y(P_2) - (x(P_2) - x)y(P_1)}{x(P_1) - x(P_2)},
\end{aligned}
\tag{5}$$

is an isomorphism (if $x(P_1) = x(P_2)$ the above definition for $v(x)$ is to be interpreted as $v(x) = \frac{dy}{dx}(P)(x - x(P)) + y(P)$). The above space of polynomials defines a smooth affine *variety of dimension two, whose coordinate ring is generated by* u_1, u_2, v_1 *and* v_2. *Therefore these can be seen as generators on the coordinate ring of* $\mathcal{A}$ *and they define meromorphic functions on* $\mathrm{Jac}(\Gamma)$; *in particular the functions* u_1 *and* u_2 *have a simple pole along* Γ_1 *and* Γ_2, *i.e.,* $u_1, u_2 \in \mathcal{L}(\Gamma_1 + \Gamma_2)$. *When* $\Gamma_1 = \Gamma_2$ *this should be understood as* $u_1, u_2 \in \mathcal{L}(2\Gamma_1)$, *i.e.,* u_1 *and* u_2 *have a double pole along* $\Gamma_1 (= \Gamma_2)$.

Proof

By the previous proposition,

$$\begin{aligned} \Gamma_i &= \tau_v\{P - \infty_i\}, \\ &= \{P - \infty_i\} + \infty_1 + \infty_2 - Q_1 - Q_2, \\ &= P + \sigma\infty_i - Q_1 - Q_2, \end{aligned}$$

and by definition the polynomials $u(x)$ and $v(x)$ associated to any point in $\mathrm{Jac}(\Gamma) \setminus (\Gamma_1 + \Gamma_2)$ satisfy $u(x) \mid f(x) - v(x)^2$ and have the right degree. Therefore the map defined by (5) is well-defined. Conversely given two such polynomials $u(x)$ and $v(x)$ we can reconstruct $x(P_1)$ and $x(P_2)$ as the roots of $u(x)$. Then evaluating $v(x)$ in $x(P_1)$ and $x(P_2)$ we find $y(P_1)$ and $y(P_2)$, thereby determining the point $\{P_1 + P_2 - Q_1 - Q_2\}$ completely. Since $x(P_1)$ and $x(P_2)$ are both finite, this point lies in $\mathrm{Jac}(\Gamma) \setminus (\Gamma_1 + \Gamma_2)$. The variety of polynomials $u(x), v(x)$, with $u(x)$ monic of degree two and $v(x)$ linear, which satisfy $u(x) \mid f(x) - v(x)^2$, is a smooth affine variety whose affine ring is generated by $\{u_1, u_2, v_1, v_2\}$. This was proved by [M] when $\deg f = 5$, the proof when $\deg f = 6$ goes exactly along the same lines. It follows that these functions are holomorphic on $\mathrm{Jac}(\Gamma)$ outside $\Gamma_1 + \Gamma_2$. Let $\{P_2 + \infty_1 - Q_1 - Q_2\}$ be any point on Γ_1 and t a local parameter around $\infty_1 \in \Gamma$. If $\Gamma_1 \neq \Gamma_2$ then $x(P) = \frac{1}{t(P)}$ for P close to ∞_1 since $f(x)$ has degree 6. Therefore u_1 and u_2 have a simple pole along Γ_1 and by symmetry also along Γ_2. If $\Gamma_1 = \Gamma_2$ then $f(x)$ has degree 5 and $x(P) = \frac{1}{t^2(P)}$ for P close to $\infty_1 = \infty_2$ in terms of a local parameter t around infinity. It follows that u_1 and u_2 have a double pole along $\Gamma_1 = \Gamma_2$. Therefore, in any case, $u_1, u_2 \in \mathcal{L}(\Gamma_1 + \Gamma_2)$. ∎

It is now our aim to calculate the functions u_1 and u_2 in terms of the original variables defining the integrable system, since then we can write down the original variables in terms of symmetric funtions on the curve. Our main tool to calculate these is based upon the existence and uniqueness of a certain singular curve in $\mathrm{Jac}(\Gamma)$, which we now establish.

Proposition 7 $\mathrm{Jac}(\Gamma)$ *contains a unique divisor* Δ, *birationally equivalent to* Γ, *with a six-fold point in* v *and smooth elsewhere.* Δ *is linearly equivalent to* $2\Gamma_1 + 2\Gamma_2$ *and is given on* $\mathcal{A} = \mathrm{Jac}(\Gamma) \setminus (\Gamma_1 + \Gamma_2)$ *by* $u_1^2 - 4u_2 = 0$, *in terms of the functions* $u_1, u_2 \in \mathcal{L}(\Gamma_1 + \Gamma_2)$ *defined in Proposition 6.*

Proof

Consider the curve Δ defined by the composition

$$\Gamma \overset{\phi}{\hookrightarrow} \Gamma \otimes_s \Gamma \overset{\mathcal{A}}{\to} \mathbb{C}^2/\Lambda \cong \mathrm{Jac}(\Gamma)$$

where ϕ is the diagonal map $\phi(P) = \langle P, P\rangle$ and A is the Abel map (2), with respect to Q_1 and Q_2. Since $\Phi = \Im(\phi)$ intersects the fundamental pencil in the points $\langle P, P\rangle$ for which $P = \sigma P$, i.e., in the six Weierstrass points, the curve $A(\phi(\Gamma))$ has a six-fold point at $\{P + \sigma P - Q_1 - Q_2\} = \{\infty_1 + \infty_2 - Q_1 - Q_2\} = v$ and is smooth elsewhere.

To find an equation for Δ remark that for a point $\{P_1 + P_2 - Q_1 - Q_2\}$ in $\mathcal{A}$, $P_1 = P_2$ iff $x(P_1) = x(P_2)$. Therefore Δ consists of the points $\{P_1 + P_2 - Q_1 - Q_2\}$ for which $(x(P_1)-x(P_2))^2 = 0$, i.e., $u_1^2-4u_2 = 0$. Since $u_1, u_2 \in \mathcal{L}(\Gamma_1+\Gamma_2)$ it follows that $u_1^2-4u_2 \in \mathcal{L}(2\Gamma_1 + 2\Gamma_2)$, so Δ is linearly equivalent to $2\Gamma_1 + 2\Gamma_2$ and the (virtual) genus of Δ is calculated from (4) as $\dim \mathcal{L}(2\Gamma_1 + 2\Gamma_2) + 1 = 17$. After desingularisation of Δ its genus equals 2 since Δ and Γ are birationally equivalent. Since in the normalisation any six-fold point acounts for a genus drop $\geq \binom{6}{2} = 15$, with equality for ordinary points only, this shows that v is an ordinary six-fold point. If there were another singular divisor Δ' of this form then Δ and Δ' would intersect in at least $6.6 = 36$ points, which is in contradiction with†

$$\Delta.\Delta' = 2(g(\Delta) - 1) = 2.16 = 32.$$

This shows that Δ is unique. ∎

If we project this curve into the Kummer surface we get a rational curve with a six-fold point. Clearly, this must be reflected in the equation of the Kummer surface. We show how the equation of the Kummer surface takes a special form and how we can find u_1 and u_2 from it — the precise choice of u_1 and u_2 will fix an affine equation of the curve (up to now only two points ∞_1 and $\infty_2 = \sigma\infty_1$ were fixed on the curve).

Proposition 8 *Let $\jmath$ be the (unique) non-trivial involution on* $\mathrm{Jac}(\Gamma)$ *which fixes v. Then $\jmath\Gamma_1 = \Gamma_2$, hence the divisor $\Gamma_1+\Gamma_2$ is invariant under $\jmath$. When $\Gamma_1 = \Gamma_2$ the restriction of $\jmath$ to Γ_1 is precisely the hyperelliptic involution on Γ_1. The involution $\jmath$ fixes sixteen points on* $\mathrm{Jac}(\Gamma)$, *six of which lie on Γ_1 when $\Gamma_1 = \Gamma_2$ (namely the six Weierstrass points on Γ_1). If u_3 is any function in $\mathcal{L}(\Gamma_1+\Gamma_2)$ such that $\{1, u_1, u_2, u_3\}$ forms a base for $\mathcal{L}(\Gamma_1+\Gamma_2)$ then the equation for an affine part $\mathcal{K}_0$ of the Kummer surface $\mathcal{K} = \mathrm{Jac}(\Gamma)/\jmath$ (the projection of $\Gamma_1 + \Gamma_2$ being removed) is given by*

$$(u_1^2 - 4u_2)u_3^2 + f_3(u_1, u_2)u_3 + f_4(u_1, u_2) = 0,$$

where f_3 and f_4 are polynomials of degree 3 and 4 respectively. Conversely, if $\{1, \theta_1, \theta_2, \theta_3\}$ forms a base for $\mathcal{L}(\Gamma_1 + \Gamma_2)$ for which the embedding of $\mathcal{K}$ in $\mathbb{P}^3$ defined by

$$\begin{aligned}\phi\colon \mathcal{K}_0 &\hookrightarrow \mathbb{P}^3\\ P &\mapsto (1 : \theta_1(P) : \theta_2(P) : \theta_3(P))\end{aligned}$$

† Formula (4) for the self-intersection of a divisor may be used for the intersection of two divisors on a generic Abelian surface since the Néron-Severi group of such a surface is isomorphic to **Z**.

sends v *to* $(0:0:0:1)$ *and for which the equation of* $\mathcal{K}_0$ *takes the form*

$$(\theta_1^2-4\theta_2)\theta_3^2+f_3'(\theta_1,\theta_2)\theta_3+f_4'(\theta_1,\theta_2)=0,$$

then θ_1 *and* θ_2 *are related to* u_1 *and* u_2 *by*

$$\begin{aligned}\theta_1&=\alpha(u_1+2\beta),\\ \theta_2&=\alpha^2(u_2+\beta u_1+\beta^2),\end{aligned}$$

for some $\alpha\in\mathbb{C}^*$, $\beta\in\mathbb{C}$ *(and conversely). This means that* θ_1 *and* θ_2 *are the functions* u_1 *and* u_2 *constructed with respect to some cover* $\pi'\colon\Gamma\to\mathbb{P}^1$, *the two covers being related by the transformation* $x\to\alpha(x-\beta)$ *of* $\mathbb{P}^1$. *The choice of* θ_1 *and* θ_2 *fixes a cover* $\pi\colon\Gamma\to\mathbb{P}^1$ *and hence the associated equation* $y^2=f(x)$ *of the curve.*

Proof

Define $\jmath_0\colon\mathrm{Jac}(\Gamma)\to\mathrm{Jac}(\Gamma)$ by $\jmath_0(\{P_1+P_2-\infty_1-\infty_2\})=\{\sigma P_1+\sigma P_2-\infty_1-\infty_2\}$, then $\jmath_0$ is a non-trivial involution which fixes $O=\{0\}$ since $\sigma\infty_1=\infty_2$. It is also an automorphism of $\mathrm{Jac}(\Gamma)$ when O is taken to be the origin; for a generic Abelian surface there is only one non-trivial automorphism of order two, hence $\jmath_0$ must lift to the reflection of the universal covering space $\mathbb{C}^2$ of the surface with respect the origin and it has sixteen fixed points, the *half-periods* of $\mathrm{Jac}(\Gamma)$, which are given by $\{B_i+B_j-\infty_1-\infty_2\}$ for all the Weierstrass points B_i on Γ (fifteen of them are given by $B_i\neq B_j$; the origin is given by $\{2B_i-\infty_1-\infty_2\}$ for any $i=1,\ldots,6$). It is easily seen that $\imath_1(\Gamma)$, which consists of the points $\{P-\infty_1\}$ is mapped to $\imath_2(\Gamma)$ since the latter curve consists of the points $\{Q-\infty_2\}$, hence $\imath_1(\Gamma)+\imath_2(\Gamma)$ is invariant under $\jmath_0$. Mumford calls such a divisor a *symmetric divisor,* see [LB]. If $\imath_1(\Gamma)=\imath_2(\Gamma)$ then $\infty_1=\infty_2$ and $\{P-\infty_1\}$ is mapped to $\{\sigma P-\infty_2\}=\{\sigma P-\infty_1\}$ so that $\jmath_0$ is exactly the hyperelliptic involution on $\imath_1(\Gamma)$. Then ∞_1 is a Weierstrass point of Γ, say $\infty_1=\infty_2=B_1$ so the six half-periods $\{B_i-\infty_1\}$ $(i=1,\ldots,6)$ of $\mathrm{Jac}(\Gamma)$ lie on $\imath_1(\Gamma)$. If we define now $\jmath=\tau_v\jmath_0\tau_{-v}$ and look at v as the origin for the group structure on $\mathrm{Jac}(\Gamma)$, we get the unique non-trivial involutive automorphism on $\mathrm{Jac}(\Gamma)$ (which fixes v) and everything which we just proved for $\imath_1(\Gamma)$ and $\imath_2(\Gamma)$ gives by translation exactly the first (three) statements in the proposition.

The main flavor of Kummer surfaces is that they can be embedded in $\mathbb{P}^3$ using the functions of $\mathcal{L}(\Gamma_1+\Gamma_2)$ and they can be described by a quartic equation. To obtain an equation for our Kummer surface $\mathrm{Jac}(\Gamma)/\jmath$ in terms of $\mathcal{L}(\Gamma_1+\Gamma_2)$ we must enlarge our set of 3 functions $\{1,u_1,u_2\}$ to a base $\{1,u_1,u_2,u_3\}$ for $\mathcal{L}(\Gamma_1+\Gamma_2)$ to find an embedding

$$\begin{aligned}\phi\colon\mathcal{K}_0&\hookrightarrow\mathbb{P}^3\\ P&\mapsto(1:u_1(P):u_2(P):u_3(P)),\end{aligned}$$

which extends to an embedding of $\mathcal{K}$. By choosing local parameters at ∞_1 and ∞_2 it can be shown as in the proof of Proposition 6 that v, viewed as a point of $\mathcal{K}$, is mapped to $(0,0,0,1)$. Since v is a double point of $\mathcal{K}$, the quartic equation of $\mathcal{K}$ reduces to

$$f_2(u_1,u_2)u_3^2+f_3(u_1,u_2)u_3+f_4(u_1,u_2)=0,\tag{6}$$

where the polynomials f_i have degree i.

Remark however that the singular divisor Δ which is birationally equivalent to Γ is invariant under $\jmath$ so that its projection $\Delta/\jmath$ in the Kummer surface is a rational curve. Since the projection has a six-fold point for $u_3 \to \infty$, (6) reduces to $P_6(u_1)u_3 + P(u_1) = 0$ for some polynomial P_6 of degree 6, when setting $u_1^2 - 4u_2 = 0$. Therefore $f_2(u_1, u_2) = u_1^2 - 4u_2$ (up to some non-zero constant) and the equation (6) of the Kummer surface reads

$$(u_1^2 - 4u_2)u_3^2 + f_3(u_1, u_2)u_3 + f_4(u_1, u_2) = 0.$$

If, on the other hand, any base $\{1, \theta_1, \theta_2, \theta_3\}$ for $\mathcal{L}(\Gamma_1 + \Gamma_2)$ is given, for which v is mapped to $(0 : 0 : 0 : 1)$ and for which the embedded Kummer surface is given by an equation

$$(\theta_1^2 - 4\theta_2)\theta_3^2 + f_3'(\theta_1, \theta_2)\theta_3 + f_4'(\theta_1, \theta_2) = 0,$$

then the functions u_i and θ_i are related by a linear transformation, in particular

$$\theta_1 = a_1u_1 + a_2u_2 + a_3,$$
$$\theta_2 = b_1u_1 + b_2u_2 + b_3,$$

for some constants $a_1, \ldots, b_3$. Since Δ is unique, $\theta_1^2 - 4\theta_2 = \alpha^2(u_1^2 - 4u_2)$ for some $\alpha \in \mathbb{C}^*$, leading to

$$\theta_1 = \alpha(u_1 + 2\beta),$$
$$\theta_2 = \alpha^2(u_2 + \beta u_1 + \beta^2),$$

for some $\alpha \in \mathbb{C}^*$, $\beta \in \mathbb{C}$. Using the definitions of u_1 and u_2 it is easy to see that

$$\theta_1(\{P_1 + P_2 - Q_1 - Q_2\}) = -(\alpha x(P_1) - \alpha\beta) - (\alpha x(P_2) - \alpha\beta)$$
$$\theta_2(\{P_1 + P_2 - Q_1 - Q_2\}) = (\alpha x(P_1) - \alpha\beta)(\alpha x(P_2) - \alpha\beta),$$

so that passing to the cover π' for which $x' = \alpha x - \alpha\beta$, the functions θ_1 and θ_2 are given by

$$\theta_1(\{P_1 + P_2 - Q_1 - Q_2\}) = -x'(P_1) - x'(P_2)$$
$$\theta_2(\{P_1 + P_2 - Q_1 - Q_2\}) = x'(P_1)x'(P_2).$$

∎

We now show that the diferential equations describing a holomorphic vector field on a principally polarised Abelian surface takes a simple form in terms of the above constructed variables μ_1, μ_2.

Theorem 9 *Suppose we are given an affine part $\mathcal{A}$ of a generic Abelian surface $\mathcal{T}^2$, which is equipped with a holomorphic which vector field $\dot{x} = X_F(x)$, and suppose $\mathcal{A}$ is principally polarised by one of the irreducible components of the divisor at infinity, $\mathcal{C}$. Denote this component of $\mathcal{C}$ by Γ_1, let v be a point on Γ_1 and let Γ_2 be the image of Γ_1 by the reflection $\jmath$ (on $\mathcal{T}^2$) which fixes v. Then for any base $\{1, u_1', u_2', u_3\}$ for $\mathcal{L}(\Gamma_1 + \Gamma_2)$*

whose associated map $\mathcal{T}^2 \to \mathbb{P}^3$ *maps* v *to* $(0:0:0:1)$, *the equation of the Kummer surface* $\mathcal{T}^2/\imath_v$ *takes the form*

$$f_2(u_1', u_2')u_3^2 + f_3(u_1', u_2')u_3 + f_4(u_1', u_2') = 0,$$

for some polynomials f_i *of degree* i; *moreover a base* $\{1, u_1, u_2, u_3\}$ *can be chosen such that the equation of this Kummer surface takes the form*

$$(u_1^2 - 4u_2)u_3^2 + f_3'(u_1, u_2)u_3 + f_4'(u_1, u_2) = 0,$$

for some polynomials f_i' *of degree* i. *Taking such a base* $\{1, u_1, u_2, u_3\}$ *and letting*

$$\begin{aligned} u_1 &= -\mu_1 - \mu_2, & X_F u_1 &= -\dot\mu_1 - \dot\mu_2 \\ u_2 &= \mu_1\mu_2, & X_F u_2 &= \mu_1\dot\mu_1 + \dot\mu_1\mu_2 \end{aligned} \tag{7}$$

the differential equations $\dot x = X_F(x)$ *can be written (explicitely) in the Jacobi form*

$$\begin{aligned} \frac{\dot\mu_1}{\sqrt{f(\mu_1)}} + \frac{\dot\mu_2}{\sqrt{f(\mu_2)}} &= \alpha_1, \\ \frac{\mu_1\dot\mu_1}{\sqrt{f(\mu_1)}} + \frac{\mu_2\dot\mu_2}{\sqrt{f(\mu_2)}} &= \alpha_2, \end{aligned}$$

for some constants α_1 *and* α_2 *(which depend on* λ*) and some equation* $y^2 = f(x)$ *for the curve* $\mathcal{C}_\lambda$. *Said differently, the roots of the polynomial* $u(x) = x^2 + u_1 x + u_2$ *are variables under which the vector field linearises. The functions* u_1 *and* u_2 *(and their derivatives) can be written down in terms of theta functions, thereby giving explicit solutions to the differential equations describing the vector field.*

Proof

The only thing which remains to be shown is that the transformation (7) reduces the differential equations to the Jacobi form.

As we saw in Proposition 6 the affine ring of $\mathcal{A} = \mathrm{Jac}(\Gamma) \setminus (\Gamma_1 + \Gamma_2)$ is generated by the functions u_1, u_2, v_1 and v_2, hence $\mathcal{A}$ is given by a set of equations in these variables. Unfortunately we cannot calculate the functions v_1 and v_2 in terms of the original phase variables, but we claim that the invariants are given by two polynomials in $u_1, u_2, \dot u_1$ and $\dot u_2$ as well. To see this we use our explicit construction of all commuting vector fields on the space of pairs of polynomials $(u(x), v(x))$ with $u(x)$ monic of degree 2 and $v(x)$ linear, satisfying $u(x) \mid f(x) - v(x)^2$ (see Section 6). We show there that for any vector field on this space which comes from a holomorphic vector field on $\mathrm{Jac}(\Gamma)$ there are constants $k, l \in \mathbb{C}$ such that

$$\begin{aligned} \dot u_1 &= kv_1 + lv_2, \\ \dot u_2 &= kv_2 + l(u_1v_2 - u_2v_1). \end{aligned}$$

Solving for v_1 and v_2 shows that the invariants are given by polynomials $G_j(u_i, \dot u_i) = 0$ as well, hence using the vector field defining the integrable system we can extend u_1 and u_2 to the above space of pairs of polynomials.

Now take any point on $\mathrm{Jac}(\Gamma_1+\Gamma_2)$ and let $\gamma(t)$ be the integral curve starting at this point. Since the vector field is holomorphic on $\mathrm{Jac}(\Gamma)$, the curve $\gamma(t)$ is given in the complex coordinates $\vec{z}=(z_1,z_2)$ coming from the natural coordinates on the universal covering space $\mathbb{C}^2$ of $\mathrm{Jac}(\Gamma)$ by $\vec{z}(t)=\vec{\alpha}t+\vec{\beta}$ for some constants $\vec{\alpha},\vec{\beta}\in\mathbb{C}^2$. Also let $\langle P_1(t),P_2(t)\rangle$ be the corresponding curve in $\Gamma\otimes_s\Gamma$ under the Abel map (2), i.e.,

$$\int_{Q_1+Q_2}^{P_1(t)+P_2(t)} \vec{\omega} \bmod \Lambda = \vec{\alpha}t+\vec{\beta} \bmod \Lambda, \tag{8}$$

where $\vec{\omega}=\binom{\omega_1}{\omega_2}$ consists of two independent holomorphic differentials on Γ, which we take here as

$$\omega_i=\frac{x^{i-1}dx}{y}=\frac{x^{i-1}dx}{\sqrt{f(x)}},$$

and Λ is the lattice of periods corresponding to ω. Taking the derivative of (8) with respect to t and writing $\mu_i(t)$ as a shorthand for $x(P_i(t))$ respectively, we get

$$\begin{aligned}\frac{\dot{\mu}_1(t)}{\sqrt{f(\mu_1(t))}}+\frac{\dot{\mu}_2(t)}{\sqrt{f(\mu_2(t))}}&=\alpha_1,\\ \frac{\mu_1(t)\dot{\mu}_1(t)}{\sqrt{f(\mu_1(t))}}+\frac{\mu_2(t)\dot{\mu}_2(t)}{\sqrt{f(\mu_2(t))}}&=\alpha_2,\end{aligned} \tag{9}$$

so $\dot{\mu}_1^2(t)$ and $\dot{\mu}_2^2(t)$ are polynomial in μ_1 and μ_2. On the other hand the integral curve can also be written down in terms of the coordinates $u_1,u_2,\dot{u}_1$ and $\dot{u}_2$ as

$$t\mapsto(u_1(t),u_2(t),\dot{u}_1(t),\dot{u}_2(t))$$

and since

$$\begin{aligned}u_1(t)&=-x(P_1(t))-x(P_2(t)),\\ &=-\mu_1(t)-\mu_2(t),\\ u_2(t)&=x(P_1(t))x(P_2(t)),\\ &=\mu_1(t)\mu_2(t),\end{aligned} \tag{10}$$

also $\dot{u}_1(t)=-\dot{\mu}_1(t)-\dot{\mu}_2(t)$ and $\dot{u}_2(t)=\dot{\mu}_1(t)\mu_2(t)+\mu_1(t)\dot{\mu}_2(t)$ so these polynomials can also be found by expressing G_j in terms of $\mu_1,\mu_2,\dot{\mu}_1$ and $\dot{\mu}_2$ using (10). The Jacobi form (9) of the differential equations as well as the equation of the curve which has (implicitely) been fixed by the choice of the functions u_1 and u_2, are deduced immediately from it. From this representation the functions $u_1(t)$ and $u_2(t)$ (and hence also the solution to the differential equations) can be written down in terms of theta functions as was shown by Mumford in [M]. ∎

The above theorem gives a systematic way to linearise general two-dimensional a.c.i. systems (whose generic invariant manifolds do not contain elliptic curves†) as given by the following theorem.

Theorem 10 *Let $\dot{x} = \{H, x\}$ be a two-dimensional a.c.i. system, as defined in [AvM1], for which the generic invariant Abelian surfaces do not contain elliptic curves. Then the vector field describing the system as well as any other vector field which commutes with this vector field can be explicitely linearised on the Jacobians of a family of hyperelliptic curves of genus two. The generic invariant Abelian surfaces are either isomorphic to this family of Jacobians or are unramified covers of these.*

Proof

If the generic affine invariant surface $\mathcal{A}_\lambda$, defined by the constants of motion of the system, is completed into an Abelian surface $\mathcal{T}^2_\lambda$ by adjoining a divisor which contains a curve Γ_λ of genus two as one of its irreducible components then the above theorem can be applied to the pair $(\text{Jac}(\Gamma), \Gamma_1) = (\mathcal{T}^2_\lambda, \Gamma_\lambda)$ for any (generic) λ. The point at infinity can either be taken as a Weierstrass point on Γ_λ in case there is only one curve of genus two at infinity, but if at least two tangent curves of genus two are found, it is more natural to pick v to be the intersection point of two of the curves.

If the (generic) Abelian surface $\mathcal{T}^2_\lambda$ is the Jacobian of a curve of genus two, but is not principally polarised by one of the irreducible components (so that all of these components have at least virtual genus 5) then the divisor at infinity is always linear equivalent to a divisor which does contain a curve of genus two, since the Néron-Sevieri group of a generic Abelian surface is isomorphic to **Z**. Therefore the above theorem can also in this case be applied after changing the divisor at infinity (i.e., taking another affine chart), a technique which will be illustrated in the Kowalevski example.

Finally, if the (generic) Abelian surface $\mathcal{T}^2_\lambda$ is not the Jacobian of a curve of genus two, then it is an n-fold cover of such a Jacobian. We stress however that neither the cover nor its order is unique (see [HvM]). Let $\pi: \mathcal{T}^2_\lambda \to \text{Jac}(\Gamma_\lambda)$ any unramified cover of order n. Then the inverse image $\mathcal{C}_\lambda = \pi^{-1}(\Theta_\lambda)$ of the Riemann theta divisor is a smooth curve in $\mathcal{T}^2_\lambda$ which is an n-fold unramified cover of Γ_λ hence it has (by the Riemann-Hurwitz formula) genus

$$g(\mathcal{C}_\lambda) = n(g(\Gamma_\lambda) - 1) + 1 = n + 1.$$

For simplcity we suppose here that the point v_λ is chosen such that Θ_λ is invariant under the involution with respect to v_λ.

To apply the previous theorem we look now for functions in $\mathcal{L}(2\mathcal{C}_\lambda)$. There will be $4n$ independent functions in $\mathcal{L}(2\mathcal{C}_\lambda)$, 4 of them can be taken to be the functions $\theta_i \circ \pi$ where $\{\theta_0 = 1, \theta_1, \theta_2, \theta_3\}$ is any base for $\mathcal{L}(2\Theta_\lambda)$. Moreover, the functions of the form $\theta \circ \pi$ where $\theta \in \mathcal{L}(2\Theta_\lambda)$ are the only functions in $\mathcal{L}(2\mathcal{C}_1)$ which are invariant under the covering transformations of the cover π. It follows that once we know these covering transformations, we can construct a base $\{1, \theta_1, \theta_2, \theta_3\}$ for $\mathcal{L}(2\Theta_\lambda)$ starting from a base for $\mathcal{L}(2\mathcal{C}_\lambda)$ which can

† When the generic invariant manifolds contain elliptic curves then solutions can be written down immediately in terms of elliptic functions.

be calculated using the Laurent solutions to the differential equations. Then we can write down an equation for the Kummer surface of $\mathrm{Jac}(\Gamma_\lambda)$ in terms of this base for $\mathcal{L}(2\Theta_\lambda)$ and hence, use the previous theorem (with $\Gamma_1 = \Gamma_2 = \Theta_\lambda$) to linearise the system and write the functions θ_i in terms of theta functions. Using the cover π, explicit solutions to the differential equations are again written down in terms of theta functions. A key example to illustrate this in the polarisation $(1,2)$ case is the Kowalevski top (see below). Also the Hénon-Heiles system is included to give an easy example in the polarisation $(1,2)$ case. ∎

4. The construction of Darboux coordinates and action-angle variables

In this section we show that starting from the Jacobi form (9) of the differential equations of a two-dimensional a.c.i. system we can determine (under some mild assumption) four new variables $(p_1, p_2, \phi_1, \phi_2)$ on a torus neighborhood of each (generic) invariant surface of the underlying (real) Liouville integrable system such that, in these coordinates, the symplectic form ω is given by $\omega = dp_1 \wedge d\phi_1 + dp_2 \wedge d\phi_2$ and the Hamiltonians defining the two flows on each invariant torus depend on p_1 and p_2 only. For an alternative approach we refer to [VN].

Since by definition the Hamiltonian vector field X_{F_i} corresponding to F_i is given by $\imath_{X_{F_i}}\omega = dF_i$, it follows that $X_{F_i}(p_j) = dp_j(X_{F_i}) = 0\,(j = 1, 2)$. Therefore p_1 and p_2 are constant along any integral curve of X_{F_i}. Secondly $X_{F_i}(\phi_j) = d\phi_j(X_{F_i}) = -\frac{\partial F_i}{\partial p_j}$ depends only on p_1 and p_2 hence ϕ_1 and ϕ_2 move linearly in time and describe exactly linear (quasi-periodic) motion on the (real) torus. Because of this property the variables ϕ_1 and ϕ_2 are classically called *angle variables,* while the conjugate variables p_1 and p_2 which are constant on each torus are called *action variables* (they have the dimension of action). We remark that the angle variables are multivalued functions, i.e., they are defined on the universal covering space of a torus neighborhood of the (real) invariant surface in exactly the same way as the usual angle variable on the circle which is in fact defined on the real line.

The construction goes in two parts; first we construct variables ν_1 and ν_2 such that, in terms of μ_1, μ_2, ν_1 and ν_2, the symplectic structure takes the canonical form $\omega = \mu_1 \wedge \nu_1 + \mu_2 \wedge \nu_2$ (such coordinates are classically called *Darboux coordinates*) and then we apply Arnold's general method to calculate the action-angle variables starting from a set of Darboux coordinates. Since these variables as well as the Darboux coordinates we construct will depend on the choice of symplectic structure and since most integrable systems have a lot of symplectic structures (as we will see in the next section and in the examples) we fix one particular symplectic structure ω and two Hamiltonian functions F_1 and F_2 which define independent vector fields on the tori (using ω). Fixing a maximal independent set of constants of motion $\{F_1, F_2, \ldots, F_k\}$ we denote for generic $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{C}^k$ by $\mathcal{A}_\lambda$ the affine invariant surface defined by $F_i = \lambda_i$. Applying the methods described in the preceeding section we are able to write the differential equations corresponding to F_1 and F_2 in the Jacobi form

$$\begin{aligned} \frac{X_{F_2}\rho_1}{\sqrt{g(\rho_1)}} + \frac{X_{F_2}\rho_2}{\sqrt{g(\rho_2)}} &= a, & \qquad \frac{X_{F_1}\rho_1}{\sqrt{g(\rho_1)}} + \frac{X_{F_1}\rho_2}{\sqrt{g(\rho_2)}} &= c, \\ \frac{\rho_1 X_{F_2}\rho_1}{\sqrt{g(\rho_1)}} + \frac{\rho_2 X_{F_2}\rho_2}{\sqrt{g(\rho_2)}} &= b, & \qquad \frac{\rho_1 X_{F_1}\rho_1}{\sqrt{g(\rho_1)}} + \frac{\rho_2 X_{F_1}\rho_2}{\sqrt{g(\rho_2)}} &= d, \end{aligned} \tag{11}$$

where g is a monic polynomial of degree 5 or 6 defining a curve Γ_λ. The constants a, b, c, d are constant on each invariant torus, but may depend on the torus itself, i.e., they may depend on λ. Since F_1 and F_2 are independent, the matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is non-singular. By a simple homographic transformation on the equation of the curve Γ_λ, this matrix becomes the identity matrix, as is shown in the following lemma.

Lemma 11. *Under the homographic transformation*

$$\mu = -\frac{a\rho - b}{c\rho - d}$$

on Γ_λ, the Jacobi form (11) for the differential equations corresponding to F_1 and F_2 is transformed into

$$\frac{X_{F_2}\mu_1}{\sqrt{f(\mu_1)}} + \frac{X_{F_2}\mu_2}{\sqrt{f(\mu_2)}} = 1, \qquad \frac{X_{F_1}\mu_1}{\sqrt{f(\mu_1)}} + \frac{X_{F_1}\mu_2}{\sqrt{f(\mu_2)}} = 0,$$
$$\frac{\mu_1 X_{F_2}\mu_1}{\sqrt{f(\mu_1)}} + \frac{\mu_2 X_{F_2}\mu_2}{\sqrt{f(\mu_2)}} = 0, \qquad \frac{\mu_1 X_{F_1}\mu_1}{\sqrt{f(\mu_1)}} + \frac{\mu_2 X_{F_1}\mu_2}{\sqrt{f(\mu_2)}} = 1,$$

$f(\mu)$ is the polynomial defined by $f(\mu) = (ad - bc)^4 g(\rho)(-c\rho + d)^{-6}$.

Proof

Substitute $\mu_i = -\frac{a\rho_i - b}{c\rho_i - d}$ in (11) and let $f(\mu) = (ad - bc)^4 g(\rho)(-c\rho + d)^{-6}$, then

$$\begin{aligned}
\sum_{j=1}^{2} \frac{X_{F_i}\mu_j}{\sqrt{f(\mu_j)}} &= (ad - bc) \sum_{j=1}^{2} \frac{X_{F_i}\rho_j}{(c\rho_j - d)^2 \sqrt{f(\mu_j)}}, \\
&= (ad - bc)^{-1} \sum_{j=1}^{2} \frac{X_{F_i}\rho_j(-c\rho_j + d)}{\sqrt{g(\rho_j)}}, \\
&= \begin{cases} 0 & i = 1, \\ 1 & i = 2. \end{cases}
\end{aligned}$$

In the same way

$$\begin{aligned}
\sum_{j=1}^{2} \frac{\mu_j X_{F_i}\mu_j}{\sqrt{f(\mu_j)}} &= (ad - bc)^{-1} \sum_{j=1}^{2} \frac{X_{F_i}\rho_j(a\rho_j - b)}{\sqrt{g(\rho_j)}}, \\
&= \begin{cases} 1 & i = 1, \\ 0 & i = 2. \end{cases}
\end{aligned}$$

∎

In the sequel we suppose that $y^2 = f(\mu)$ is the (unique) equation of the curve given by the previous lemma. Also we will *assume* that the coefficients of ρ in f, which depend on the constants of motion λ_i, depend polynomially on λ_1 and λ_2, the constants corresponding to the independent Hamiltonians F_1 and F_2. This is true in all examples we have checked — it might be a theorem. We show that under this assumption the symplectic structure takes a simple form in the coordinates μ_1, μ_2, F_1 and F_2.

Proposition 12 *The symplectic structure ω is given by*

$$\omega=\sum_{i=1}^{2}\frac{\mu_i d\mu_i\wedge dF_1+d\mu_i\wedge dF_2+dG(F_1)\wedge dF_2}{\sqrt{f(\mu_i)}}\tag{12}$$

for some function G depending on F_1 only.

Proof

Since ω vanishes when applied to any pair of vectors tangent to the tori $F_1, F_2 =$ constant, it takes the form $\omega=\omega_1\wedge dF_1+\omega_2\wedge dF_2+\chi dF_1\wedge dF_2$, for some one-forms ω_1 and ω_2 and some function χ. Since in an a.c.i. system the Hamiltonian vector fields are holomorphic when restricted to the invariant Abelian surfaces (tori) and since $\imath_{X_{F_1}}\omega=dF_1$, and $\imath_{X_{F_2}}\omega=dF_2$, imply $\omega_i(X_{F_j})=\delta_{ij}$, the restriction of the one-forms ω_1 and ω_2 to the invariant tori are holomorphic as well. Now a base for the holomorphic one forms on the Jacobian of the curve $y^2=f(\mu)$ is given by $\{\Omega_1,\Omega_2\}$, where

$$\Omega_i=\sum_{j=1}^{2}\frac{\mu_j^{i-1}d\mu_j}{\sqrt{f(\mu_j)}},\qquad (i=1,2),$$

so that $\omega_i=A_i\Omega_1+B_i\Omega_2$ $(i=1,2)$, where A_1,A_2,B_1 and B_2 may a priori depend on F_1 and F_2. Since $\Omega_i(X_{F_j})=\delta_{ij}$, it follows that $\omega_1=\Omega_2$ and $\omega_2=\Omega_1$, giving

$$\omega=\Omega_2\wedge dF_1+\Omega_1\wedge dF_2+\chi dF_1\wedge dF_2.$$

We conclude the proof by showing that the function χ takes a special form. Since ω is closed,

$$\frac{1}{f(\mu_i)\sqrt{f(\mu_i)}}\left(\mu_i\frac{\partial f}{\partial F_2}(\mu_i)-\frac{\partial f}{\partial F_2}(\mu_i)\right)-2\frac{\partial\chi}{\partial\mu_i}=0,$$

for $i=1,2$. Since μ_1 and μ_2 are symmetric coordinates, χ is symmetric in μ_1 and μ_2 it follows from this equation that $\chi=\psi(\mu_1)+\psi(\mu_2)$ (the dependence of ψ on F_1 and F_2 is ommitted in the notation). It follows that $\psi(\mu)$ has a derivative $\frac{P(\mu)}{f(\mu)\sqrt{f(\mu)}}$ for some polynomial $P(\mu)$ where $\deg P<\deg f$. Together with the fact that $\psi(\mu)$ is algebraic it follows that $\psi(\mu)$ is of the form $\psi(\mu)=\frac{Q(\mu)}{\sqrt{f(\mu)}}$ for some polynomial Q; differentiating this function and using $\deg P<\deg f$ one finds that $\deg Q<1$, i.e., Q depends only on F_1 and F_2, and

$$\chi=\left(\frac{1}{\sqrt{f(\mu_1)}}+\frac{1}{\sqrt{f(\mu_2)}}\right)\phi(F_1,F_2)$$

for some function ϕ of F_1 and F_2.

Next we show that ϕ is independent of F_2 in case the degree of f equals 5, the case $\deg(f)=6$ being very similar. Note that in terms of ϕ the closedness of ω simply reads

$$\mu\frac{\partial f}{\partial F_2}(\mu)-\frac{\partial f}{\partial F_1}(\mu)+\phi\frac{\partial f}{\partial\mu}(\mu)=0.\tag{13}$$

Writing f as $f(\mu) = \mu^5 + A_1\mu^4 + A_2\mu^3 + A_3\mu^2 + A_4\mu + A_5$, where $A_1, \ldots, A_5$ are by assumption polynomials in F_1 and F_2, this condition reduces to

$$\begin{aligned}
\frac{\partial A_1}{\partial F_2} &= 0, & \frac{\partial A_4}{\partial F_2} - \frac{\partial A_3}{\partial F_1} + 3\phi A_2 &= 0,\\
\frac{\partial A_2}{\partial F_2} - \frac{\partial A_1}{\partial F_1} + 5\phi &= 0, & \frac{\partial A_5}{\partial F_2} - \frac{\partial A_4}{\partial F_1} + 2\phi A_3 &= 0,\\
\frac{\partial A_3}{\partial F_2} - \frac{\partial A_2}{\partial F_1} + 4\phi A_1 &= 0, & -\frac{\partial A_5}{\partial F_1} + \phi A_4 &= 0.
\end{aligned}$$

From the second equation it follows that ϕ is a polynomial in F_1 and F_2. Suppose that ϕ depends on F_2, $\deg_{F_2}(f) = j \geq 1$. The coefficient of F_2^j in f is a polynomial in F_1, say of degree $i \geq 0$. Thus ϕ contains a term $F_1^i F_2^j$. We call this the *top term* of ϕ. By the first equation A_1 is independent of F_2. Then the second equation implies that A_2 and $A_2\phi$ have as top term $F_1^i F_2^{j+1}$ and $F_1^{2i} F_2^{2j+1}$ respectively. Then it follows from the fourth equation that A_4 has as top term $F_1^{2i} F_2^{2j+2}$, since if $\frac{\partial A_3}{\partial F_1}$ contains $F_1^{2i} F_2^{2j+2}$ then $\frac{\partial A_3}{\partial F_2}$ contains $F_1^{2i+1} F_2^{2j+1}$, which is incompatible with the third equation. It follows from the last equation that A_5 has as top term $F_1^{3i+1} F_2^{3j+2}$. Finally, using the fifth equation, this would force A_3 to contain a term in $F_1^{2i+1} F_2^{2j+1}$ which is incompatible with the third equation. This shows that $j \geq 1$ is impossible, so that ϕ does not depend on F_2. ∎

In concrete examples the above form for the symplectic structure can be found by expressing the original symplectic structure in terms of μ_1, μ_2, F_1 and F_2 by using the linearising map. It is well known that such a calculation is very tedious. The preceeding proposition however gives an extremely simple and useful method to write the symplectic structure in this form; the only thing to be determined in (12) is the function $G(F_1)$, and this function is just a primitive of $\phi(F_1)$ which is found immediately from the equation of the curve by using (13); moreover in all examples except one (the Kowalevski top) it will turn out that

$$\mu \frac{\partial f}{\partial F_2}(\mu) - \frac{\partial f}{\partial F_1}(\mu) = 0.$$

so that $G(F_1) = 0$ and the symplectic structure is found immediately to have a very pretty form in these coordinates. We show in the following proposition that Darboux coordinates can be obtained easily from Proposition 12.

Proposition 13 *Letting $H(F_1)$ be a primitive of $G(F_1)$, $\rho_i = \mu_i + G(F_1)$ and $\Delta_i = F_2 + \rho_i F_1 - H(F_1)$, the symplectic structure ω can be written as*

$$\omega = \sum_{i=1}^{2} \frac{d\rho_i \wedge d\Delta_i}{\sqrt{g(\rho_i)}},$$

where $g(\rho_i)$ is the monic polynomial $f(\rho_i - G(F_1))$. Also $g(\rho_i)$ depends on ρ_i and Δ_i only (instead of depending on ρ_i, F_1 and F_2), which we sometimes stress by writing $g(\rho_i)$ as

$g(\rho_i, \Delta_i)$. *Therefore* ρ_1, ρ_2, σ_1 *and* σ_2 *are Darboux coordinates,* $\omega = d\rho_1 \wedge d\sigma_1 + d\rho_2 \wedge \sigma_2$, *where* σ_i *is a primitive for* $\frac{d\Delta_i}{\sqrt{g(\rho_i,\Delta_i)}}$ *(keeping* ρ_i *fixed when integrating).*

Proof

By Proposition 12,

$$\begin{aligned}\omega &= \sum_{i=1}^{2} \frac{\mu_i d\mu_i \wedge dF_1 + d(\mu_i + G(F_1)) \wedge dF_2}{\sqrt{f(\mu_i)}}, \\ &= \sum_{i=1}^{2} \frac{\mu_i d(\mu_i + G(F_1)) \wedge dF_1 + d(\mu_i + G(F_1)) \wedge dF_2}{\sqrt{f(\mu_i)}}.\end{aligned}$$

This suggests setting $\rho_i = \mu_i + G(F_1)$; also let $g(\rho_i) = f(\rho_i - G(F_1))$ and let $H(F_1)$ be a primitive of $G(F_1)$. Then

$$\begin{aligned}\omega &= \sum_{i=1}^{2} \frac{(\rho_i - G(F_1))d\rho_i \wedge dF_1 + d\rho_i \wedge dF_2}{\sqrt{g(\rho_i)}}, \\ &= \sum_{i=1}^{2} \frac{d\rho_i \wedge d(\rho_i F_1 + F_2 - H)}{\sqrt{g(\rho_i)}}, \\ &= \sum_{i=1}^{2} \frac{d\rho_i \wedge d\Delta_i}{\sqrt{g(\rho_i)}},\end{aligned} \tag{14}$$

where Δ_i is a shorthand for $\rho_i F_1 + F_2 - H(F_1)$.

The coefficients of μ in $g(\mu)$ depend on F_1 and F_2, or equivalently, since F_2 appears linearly in Δ_i, on F_1 and Δ_i. Since ω is closed it follows from the last expression in (14) that $g(\rho_i, \Delta_i, F_1)$ does not depend explicitely on F_1. Therefore we may integrate to obtain

$$\sigma_i = \int \frac{d\Delta_i}{\sqrt{g(\rho_i, \Delta_i)}}$$

which puts ω in the canonical form $\omega = \sum d\rho_i \wedge d\sigma_i$. ∎

Remark

In most cases (in all cases we have seen) the equation $y^2 = g(\rho_i, \Delta_i)$ depends at worst quadratically on Δ_i. It follows that σ_i can be expressed in terms of elementary functions (which are always trancendental).

It was shown by Arnold how to find explicitely the action-angle variables starting from Darboux coordinates. We recall his construction here, adapted to the case of two-dimensional a.c.i. systems. As was shown in [F], it turns out that we can even be more explicit in this case: the one-forms and the cycles on the Abelian surface can be translated into one-forms and cycles on the (real) curve. We refer to [A] and [AM] for a proof of Arnold's construction which we recall now.

Arnold constructs action-angle variables in the neighborhood of a (generic) invariant manifold of a (real) Liouville integrable system; there is an obstruction to construct them globally as was first shown in a beautiful paper by Duistermaat [D]. In our case the (real) invariant manifolds are two-dimensional tori, diffeomorphic to $S^1 \times S^1$, and the neighborhood is taken to be a torus neighborhood, i.e., diffeomorphic to a product of such a torus with a disc. We will for $c \in \mathbb{R}^2$ denote by I_c the (real) invariant manifold $F^{-1}(c) = (F_1, F_2)^{-1}(c)$. Fixing a generic c_0, let U be a neighborhood of c_0 such that $F^{-1}(U)$ is a torus neighborhood. By (local) exactness of the (closed) symplectic form, $\omega = -d\theta$ on $F^{-1}(U)$ with $\theta = \sum \sigma_i d\rho_i$. Next we need to choose a base $\{\gamma_1(c), \gamma_2(c)\}$ for $H_1(I_c, \mathbf{Z}) \cong \mathbf{Z}^2$ varying continuously with $c \in U$. Since the homology of the curve generates the homology of its Jacobian (by the *Lefschetz hyperplane theorem*) we may choose a consistent base for the homology of the family of curves corresponding to the tori. Since we are dealing with real Jacobians here, at least four of the Weierstrass points of the curve are real, and two cycles can be taken to be intervals on the real axis, say $[e_1(c), e_2(c)]$ and $[e_3(c), e_4(c)]$, and we take $\gamma_1(c)$ and $\gamma_2(c)$ to be the corresponding cycles on the surface. Then the integrals

$$P_j(c) = \oint_{\gamma_j(c)} \theta$$

are well-defined ($j = 1, 2$) and using Proposition 13 they are given by integrating some trancendental differential form on the curve:

$$P_i(c) = 2 \int_{e_{2i-1}}^{e_{2i}} \left[\int \frac{d\Delta}{\sqrt{g(\rho, \Delta)}} \right] d\rho$$

(we could drop the index j). The functions $p_1 = P_1 \circ F$ and $p_2 = P_2 \circ F$ are by definition constant on each torus and have the dimension of action. By shrinking U if necessary, we may assume that the image $\mathrm{im}(p_1, p_2) \subset \mathbb{R}^2$ is a disc D^2. Extending the variables p_1 and p_2 to a set of Darboux coordinates $\{p_1, p_2, \phi_1, \phi_2\}$ (in the classical way, namely using a canonical transformation $(\rho_i, \sigma_i) \to (\phi_i, p_i)$) the variation of ϕ_i on the cycle γ_j is exactly δ_{ij} and the invariants depend on p_1 and p_2 only. Therefore these canonical variables, defined on a torus neighborhood are action-angle variables. We will calculate these variables in some examples in the next section.

5. Compatible Poisson brackets for integrable systems

It was first observed by Lenard and Magri and later by others that a lot of integrable systems possess different symplectic structures, or more generally, different Poisson brackets (see [Ma]). The basic vector field defining the system appears to be Hamiltonian for both structures, of course corresponding to different Hamiltonians. Moreover, these structures turned out to be compatible in a sense to be defined below, which leads to a lot of nice geometrical ideas and constructions, especially when one of the Poisson brackets is invertible. In the examples we found a lot of compatible Poisson brackets, which are of a simple but rather unusual form. The aim of this section is to show that for such pairs of Poisson brackets, compatibility follows at once from the form of the brackets.

Let $(P, \{\cdot,\cdot\}_1)$ be a Poisson manifold and let $\{\cdot,\cdot\}_2$ be another Poisson structure on P. Suppose that $X_F = \{\cdot, F\}_1$ defines an integrable system on P and that every vector field $X_H = \{\cdot, H\}$ which commutes with X_F, including X_F itself, is Hamiltonian for $\{\cdot,\cdot\}_2$ as well. Then we call this integrable system a *completely bi-Hamiltonian (integrable) system.* The Poisson brackets $\{\cdot,\cdot\}_1$ and $\{\cdot,\cdot\}_2$ are called *compatible* if $\{\cdot,\cdot\}_1 + \{\cdot,\cdot\}_2$ is also a Poisson bracket on P, or equivalently if any linear combination of $\{\cdot,\cdot\}_1$ and $\{\cdot,\cdot\}_2$ is a Poisson structure on P, or equivalently if the Jacobi identity is satisfied for $\{\cdot,\cdot\}_1 + \{\cdot,\cdot\}_2$ (or for any linear combination).

We show now that the Poisson brackets of a two-dimensional completely bi-Hamiltonian system are always compatible. We conjecture that the same is true in higher dimensions.

Theorem 14 *Let $(P, \{\cdot,\cdot\}_1, \{\cdot,\cdot\}_2, H)$, be a completely bi-Hamiltonian integrable system of dimension two. Then $\{\cdot,\cdot\}_1$ and $\{\cdot,\cdot\}_2$ are compatible.*

Proof

For any $f \in C^\infty(P)$, denote $X_f = \{\cdot, f\}_1$ and by $Y_f = \{\cdot, f\}_2$. Letting $\{\cdot,\cdot\} = \{\cdot,\cdot\}_1 + \{\cdot,\cdot\}_2$ it suffices to check the Jacobi identity of $\{\cdot,\cdot\}$ for functions $f, g, h \in \{H_1, \ldots, H_k, H_{k+1}, H_{k+2}\}$, where $H_1, \ldots, H_k$ generates the algebra of invariant functions of the system and H_{k+1} and H_{k+2} are any two (locally defined) functions such that $\{H_1, \ldots, H_k, H_{k+1}, H_{k+2}\}$ are independent (or, equivalently, the differentials of these functions are independent). In terms of $\{\cdot,\cdot\}_1$ and $\{\cdot,\cdot\}_2$ the Jacobi identity reduces to

$$\begin{aligned}\{\{f,g\}_1,h\}_2 + &\{\{g,h\}_1,f\}_2 + \{\{h,f\}_1,g\}_2 + \\ &\{\{f,g\}_2,h\}_1 + \{\{g,h\}_2,f\}_1 + \{\{h,f\}_2,g\}_1 = 0,\end{aligned}$$

by using the Jacobi identity for $\{\cdot,\cdot\}_1$ and $\{\cdot,\cdot\}_2$. This can also be written as

$$[X_f, Y_g] - [X_g, Y_f] + X_{\{f,g\}_2} + Y_{\{f,g\}_1} = 0.$$

Using the fact that any two invariant functions of the system also Poisson commute for the second bracket, $\{\cdot,\cdot\}_2$, we see that the Jacobi identity is valid for $f, g \in \{H_1, \ldots, H_k\}$, $h \in \{H_1, \ldots, H_{k+2}\}$. Since the Jacobi identity for $f, g, h \in \{H_{k+1}, H_{k+2}\}$ follows from antisymmetry, it suffices to check it for $f \in \{H_1, \ldots, H_k\}$, $g = H_{k+1}$, $h = H_{k+2}$. By assumption $\{\cdot, h\}_1 = \{\cdot, \bar{h}\}_2$ and $\{\cdot, h\}_2 = \{\cdot, \tilde{h}\}_1$ for some functions $\bar{h}$ and $\tilde{h}$. Therefore the Jacobi identity reduces to

$$\{\{f,g\}_1,\tilde{h}\}_1 + \{\{g,\bar{h}\}_2,f\}_2 + \{\{\bar{h},f\}_2,g\}_2 + \\ \{\{f,g\}_2,\bar{h}\}_2 + \{\{g,\tilde{h}\}_1,f\}_1 + \{\{\tilde{h},f\}_1,g\}_1 = 0,$$

which reduces to the Jacobi identity for $\{\cdot,\cdot\}_1$ and $\{\cdot,\cdot\}_2$. ∎

The completely bi-Hamiltonian structures which we found, were in fact discovered in a very systematic way, which we now describe. The idea comes from the preceeding section: we have seen that the functions F_1 and F_2 which can appear as Hamiltonians defining respectively the highest and lowest flow for some equation of the curve, must appear in this equation in the form $F_2 + \mu F_1$ (possibly after some translation in μ which can be calculated from the equation of the curve). Therefore it is natural to look for a Poisson structure which generates the two vector fields upon using the gradients of F_1 and F_2, and has a complementary set $F_3, \ldots, F_k$ of invariants as Casimirs, when the equation of the curve depends on $F_2 + \mu_1 F_1$ only, for constant values of $F_3, \ldots, F_k$. This information suffices to determine the Poisson brackets since we look for polynomial Poisson brackets.

As an example to illustrate this, suppose the equation of the curve is given by $y^2 = x^6 + ax^4 + bx^3 + cx^2 + dx + e$, where $a, \ldots, e$ generate the invariant polynomials, then one can take a as Hamiltonian for the first vector field, b for the second one, and take $c, \ldots, e$ as Casimirs. However, a and b can be replaced by b, c or c, d or d, e, the other basic invariants always being taken as Casimirs. To find the Poisson structure (for the first choice, for example) one constructs the matrix J which satisfies

$$\begin{aligned} \dot{x} &= J\nabla a(x), \\ x' &= J\nabla b(x), \\ 0 &= J\nabla c(x), \\ 0 &= J\nabla d(x), \\ 0 &= J\nabla e(x), \end{aligned}$$

where $\dot{x}$ and x' denote the two vector fields defining the system. The matrix J contains the Poisson brackets as its entries and the inverse of any non-singular 4×4 minor gives an expression for the symplectic structure on the (generic) symplectic leaves. By the above theorem all Poisson structures which are constructed in this way are compatible.

6. Examples

This section is entirely devoted to the study of (seven) examples of two-dimensional a.c.i. systems (one is a.c.i. in the generalised sense). There are several reasons for an extensive study of these examples. First, and most importantly, the interest of systematic procedures to linearise a.c.i. systems and to find action-angle variables lies not only in the beautiful geometry involved in it, but also in their applicability. Secondly, studying concrete examples has always been the major source of new ideas for understanding mechanical systems; in our case the master systems and the Kowalevski top were among the most inspirational and stimulating ones. Finally we want to show that our methods fit naturally in the systematic study of a.c.i. systems initiated by Adler and van Moerbeke (see [AvM1]), which is best done on concrete examples. Also it gives us the possibility to show on an example how the methods naturally generalise to systems which are a.c.i. in the generalised sense (see the quartic potential), to present a new a.c.i. system (the even master system) and to show how curves in an Abelian surface can be studied using an embedding of the surface in projective space, for example we show how to determine the nature of the curves corresponding to Δ in the case where the tori do not carry a principal polarisation (see the Kowalevski and Hénon-Heiles examples). These divisors, as well as the divisors to be adjoined to the affine invariant manifolds to obtain a complete variety, will be represented by schematic drawings which suggest the singularities of the divisors.

a. The three body Toda lattice

The three body Toda lattice consists of three particles interconnected by means of exponential springs and constrained to move on a circle. The motion is determined by the Hamiltonian

$$H = \frac{1}{2}\sum_{k=1}^{3} p_k^2 + \sum_{k=1}^{3} e^{q_k - q_{k+1}}, \qquad q_4 = q_1,$$

with the standard symplectic structure. Using Flaschka's change of variables $t_k = e^{q_k - q_{k+1}}$, $t_{k+3} = -p_k$, $(k = 1, \dots, 3)$, the equations of motion have the following nice form

$$\begin{aligned} \dot t_1 &= t_1(t_5 - t_4), & \dot t_4 &= t_1 - t_3,\\ \dot t_2 &= t_2(t_6 - t_5), & \dot t_5 &= t_2 - t_1,\\ \dot t_3 &= t_3(t_4 - t_6), & \dot t_6 &= t_3 - t_2. \end{aligned} \tag{15}$$

The Toda flow has the following 4 constants of motion,

$$\begin{aligned} T_1 &= t_1 t_2 t_3 = 1,\\ T_2 &= t_4 + t_5 + t_6 = 0,\\ T_3 &= \frac{1}{2}(t_4^2 + t_5^2 + t_6^2) + t_1 + t_2 + t_3 = a_1,\\ T_4 &= t_4 t_5 t_6 - t_1 t_6 - t_2 t_4 - t_3 t_5 = b_1, \end{aligned} \tag{16}$$

where we shifted t_4, t_5 and t_6 by a constant, so as to obtain $T_2 = 0$. If we assign t_1, t_2 and t_3 weight 2 and t_4, t_5 and t_6 weight 1, then the invariants are all weight homogeneous with weights 6, 1, 2 and 3 respectively. If we give time weight -1 then the first vector field also becomes weight homogeneous. Such a vector field is called *weight homogeneous;* it is shown in [AvM1] that for such a vector field it is easy to find the Laurent solutions to the differential equations.

Remark also that the invariant manifold defined by (16) has an automorphism of order 3 given by

$$(t_1, t_2, t_3, t_4, t_5, t_6) \mapsto (t_2, t_3, t_1, t_5, t_6, t_4). \tag{17}$$

This automorphism simplifies the Painlevé analysis, which applied to this system, gives three families of Laurent solutions depending on the maximal number of free parameters (5 in this case). Such families are called *principal balances.* We display one principal balance, the other two are found using the order three automorphism ($a, b, \ldots$ will always denote the free parameters entering in the balances):

$$\begin{aligned}
t_1 &= -\frac{1}{t^2}(1 + ct^2 + dt^3 + \cdots), & t_4 &= \frac{1}{t}(1 + at - ct^2 - \frac{1}{2}(5d + e)t^3 + \cdots),\\
t_2 &= -t(e + (b - a)et + \cdots), & t_5 &= \frac{1}{t}(-1 + at + ct^2 + \frac{1}{2}(d - e)t^3 + \cdots),\\
t_3 &= -t(-(4d + e) + (4d + e)(b - a)t + \cdots), & t_6 &= b + (e + 2d)t^2 + \frac{4}{3}d(a - b)t^3 + \cdots.
\end{aligned}$$

The Laurent solutions make it easy to find an embedding of the affine invariant surfaces into projective space; in our case the functions which behave like $\frac{1}{t}$ at worst when any of the Laurent solutions are substituted in them define a projective embedding of the affine invariant surface, whose closure is a principally polarised Abelian surface. A basis for these functions is given by $\{1, t_5, t_6, t_4t_5 - t_1, t_5t_6 - t_2, t_2t_4 + t_5(t_4t_5 - t_1), t_1t_2, t_2t_3, t_3t_1\}$. Three hyperelliptic curves of genus two, which can be calculated by substituting the Laurent solutions in the invariants, need to be adjoined to the affine surface to get the complete variety. They are all seen to be isomorphic to the curve

$$e^2 + e(8a^3 - 2aa_1 + b_1) + 1 = 0, \tag{18}$$

and using the embedding it is checked that any two of these curves touch in one point; also the curves are permuted by the order three involution (17). The three intersection points are also the points of tangency of the Toda flow. Representing each of the curves by a circle, this divisor at infinity can be depicted as follows.

Figure 1

We proceed to the linearisation of the Toda lattice. Three involutions leaving the divisor at infinity invariant are found by composing the involution $\jmath(t_1,t_2,t_3,t_4,t_5,t_6) = (t_1,t_3,t_2,t_5,t_4,t_6)$ with the order three automorphism (17). $\jmath$ itself fixes one of the tangency points which we call v and interchanges the two curves (which we call Γ_1 and Γ_2) of the divisor at infinity v belongs to, while it maps the other curve of this divisor to itself. Clearly the point v and the curves are taken as the basic ingredients to apply the methods explained in Section 3. Remark that although one of the divisors is mapped to itself under $\jmath$ we cannot take this curve as $\Gamma_1 = \Gamma_2$ (in the terminology of Section 3) since we need to know the coordinates in projective space of some fixed point of $\jmath$ on the curve, and this reduces to the calculation of one of the Weierstrass points on the curve (infinity is not a Weierstrass point). Since the Laurent solutions above correspond to the curve which is fixed under $\jmath$, a base for $\mathcal{L}(\Gamma_1+\Gamma_2)$ is found by looking for the functions generated by the embedding, whose Laurent expansions for this balance do not blow up. Such a base is found immediately to be given by the four basic functions $\{1, \theta_1 = -t_6, \theta_2 = t_4t_5 - t_1, \theta_3 = t_2t_3\}$. The invariants can be expressed in terms of these functions, giving a quartic equation which we identified as the equation defining the Kummer surface of the invariant manifold as a surface in $\mathbb{P}^3$. The equation reads

$$\theta_3^2(\theta_1^2 - 4\theta_2) + \theta_3(\theta_2^3 - a_1\theta_2) + P_4(\theta_1,\theta_2) = 0,$$

for some polynomial P_4 of degree 4. Remark that we have choosen $\{1,\theta_1,\theta_2,\theta_3\}$ such that v is mapped to $(0:0:0:1) \in \mathbb{P}^3$, as in Theorem 9. By this theorem the vector field defining the Toda lattice linearises upon setting

$$\begin{aligned} t_6 &= \mu_1 + \mu_2 \qquad & t_4t_5 - t_1 &= \mu_1\mu_2, \\ t_3 - t_2 &= \dot\mu_1 + \dot\mu_2 \qquad & t_2t_4 - t_3t_5 &= \dot\mu_1\mu_2 + \mu_1\dot\mu_2. \end{aligned} \tag{19}$$

Substituting (19) in the invariants (16) and eliminating the other variables, two polynomials in $\dot\mu_1^2$ and $\dot\mu_2^2$ are found, one is linear, the other is quadratic. Solving them for $\dot\mu_i^2$ yields

$$\dot\mu_i^2 = \frac{\mu_i^6 - 2\mu_i^4a_1 + 2\mu_i^3b_1 + \mu_i^2a_1^2 - 2\mu_ia_1b_1 - 4 + b_1^2}{(\mu_1-\mu_2)^2},$$

which leads immediately to

$$\begin{aligned} \frac{d\mu_1}{\sqrt{f(\mu_1)}} + \frac{d\mu_2}{\sqrt{f(\mu_2)}} &= 0, \\ \frac{\mu_1 d\mu_1}{\sqrt{f(\mu_1)}} + \frac{\mu_2 d\mu_2}{\sqrt{f(\mu_2)}} &= 1, \end{aligned}$$

where f is the polynomial $f(\mu) = (\mu^3 - a_1\mu + b_1)^2 - 4$. Clearly the curve $y^2 = f(x)$ is isomorphic to the curve (18). In the same way it is shown that the vector field corresponding to $-T_4$ gives the highest flow with respect to the same equation $y^2 = f(x)$ for the curve. This completes the linearisation of the Toda lattice.

Finally we look at the symplectic structure for the Toda lattice and deduce Darboux coordinates as well as action-angle variables on a torus neighborhood of a generic invariant torus of the real system (see [FM]). Using Flaschka's change of variables and our normalisation $t_4+t_5+t_6=0$ the original (standard) symplectic structure is expressed immediately in terms of the variables t_i as $\omega=\frac{dt_3\wedge dt_4}{t_3}-\frac{dt_2\wedge dt_5}{t_2}$. Since we know that T_3 and $-T_4$ give the lowest and the highest flow respectively we write $f(\mu)$ as $f(\mu)=(\mu^3-T_3\mu+T_4)^2-4$, and we remark that

$$\frac{\partial f}{\partial T_3}(\mu)=-\mu\frac{\partial f}{\partial T_4}(\mu).$$

Letting $\Delta_i=T_3\mu_i-T_4$, it follows from Proposition 12 that $\omega=d\mu_1\wedge d\nu_1+d\mu_2\wedge d\nu_2$, where

$$\nu_i=\int\frac{d\Delta_i}{\sqrt{(\mu_i^3-\Delta)^2-4}}=\log|\Delta_i-\mu_i^3+\sqrt{(\mu_i^3-\Delta_i)^2-4}|. \qquad (i=1,2)$$

Applying Arnold's method, action variables are given by

$$p_i=2\int_{e_{2i-1}}^{e_{2i}}\log|\Delta-\mu^3+\sqrt{(\mu^3-\Delta)^2-4}|d\mu \tag{20}$$

the points $e_1,\ldots,e_4$ being four (real) branch points on the curve $y^2=(\mu^3-\Delta)^2-4$; the fact that they depend on the constants of motion is dropped in the notation. Let $S(\mu_1,\mu_2,T_3,T_4)$ be the function defined by

$$S=\int_{e_1}^{\mu_1}\log|\Delta-\mu^3+\sqrt{(\mu^3-\Delta)^2-4}|d\mu+\int_{e_3}^{\mu_2}\log|\Delta-\mu^3+\sqrt{(\mu^3-\Delta)^2-4}|d\mu;$$

S is the generating function of a canonical transformation $(\mu_1,\mu_2,\nu_1,\nu_2)\to(\phi_1,\phi_2,p_1,p_2)$, so that angle variables are found by

$$\phi_i=\frac{\partial S}{\partial p_i}=\frac{\partial S}{\partial T_3}\frac{\partial T_3}{\partial p_i}+\frac{\partial S}{\partial T_4}\frac{\partial T_4}{\partial p_i},$$

where $\frac{\partial S}{\partial T_3}$ and $\frac{\partial S}{\partial T_4}$ are found by direct differentiation under the integral sign. $\frac{\partial T_3}{\partial p_i}$ and $\frac{\partial T_4}{\partial p_i}$ are found by calculating $\frac{\partial p_i}{\partial T_j}$ from (20) for $i=1,2$ and $j=3,4$.

b. A seven-dimensional system

Next, we consider a seven-dimensional system constructed by Bechlivanidis and van Moerbeke when studying a top, known as the Goryachev-Chaplygin top. We refer to [BvM] for the description of this top and the precise relation with this seven-dimensional system. We use coordinates which differ only slightly from the ones in [BvM], (this makes the Laurent solutions a bit easier):

$$\begin{aligned}
\dot s_1&=-8s_7, & \dot s_4&=-4s_2s_5-s_7,\\
\dot s_2&=4s_5, & \dot s_5&=s_6-4s_2s_4,\\
\dot s_3&=2(s_4s_7+s_5s_6), & \dot s_6&=-s_1s_5+2s_2s_7,
\end{aligned} \tag{21}$$

$$\dot s_7=s_1s_4+2s_2s_6-4s_3.$$

There are five constants of motion, namely

$$
\begin{aligned}
S_1 &= s_1 - 4s_2^2 - 8s_4 = a_2, \\
S_2 &= s_1 s_2 + 4s_6 = b_2, \\
S_3 &= s_4^2 - s_5^2 + s_3 = c_2, \\
S_4 &= s_4 s_6 + s_5 s_7 + s_2 s_3 = d_2, \\
S_5 &= -s_6^2 + s_7^2 - s_1 s_3 = e_2.
\end{aligned}
\tag{22}
$$

This system is again weight homogeneous with weight 1 for s_2, 2 for s_1, s_4 and s_5, 3 for s_6, s_7 and 4 for s_3; the invariants S_i have weighted degree $i+1$. There are two principal balances depending on six free parameters, distinguished by $\epsilon = \pm 1$. The first terms of the Laurent solutions for s_1, s_2 and s_3 are given by

$$
\begin{aligned}
s_1 &= \frac{1}{t}\left(a - \frac{a^2 t}{2} + ct^2 - \frac{a}{2}(c + 4d + 2ab)t^3 + \cdots\right), \\
s_2 &= \frac{\epsilon}{t}\left(-\frac{1}{2} - \frac{at}{4} + bt^2 + dt^3 + et^4 + \cdots\right), \\
s_3 &= -\frac{1}{32t}(2ab + c + 8d) + \frac{3}{64}(ac + 8ad + 2a^2 b) + \cdots.
\end{aligned}
\tag{23}
$$

The affine surface defined by the five quadrics (22) can be embedded in projective space by means of the sixteen functions

$$
\{1, s_1, \ldots, s_7, s_1^2, s_1 s_3, s_2 s_3, s_3^2, \dot{s}_3, \{s_1, s_2\}, \{s_1, s_3\}, \{s_2, s_3\}\},
\tag{24}
$$

which behave like t^{-2} at worst when the series (23) are substituted into them ($\{s_i, s_j\}$ denotes the *Wronskian* $\dot{s}_i s_j - s_i \dot{s}_j$ of s_j and s_i). Using the embedding (24) and the series (23), it can be shown that the closure of the image of this affine surface is (for generic values of the constants of motion) an Abelian variety. This is done by adjoining two isomorphic hyperelliptic curves Γ_ϵ of genus 2. An equation for these curves is found by putting the Laurent solutions for the principal balances displayed above in the invariants (22), giving

$$
c'^2 = (a^3 + a_2 a + 2b_2\epsilon)^2 - 64(c_2 a^2 - 2a\epsilon d_2 - e_2),
\tag{25}
$$

when setting $c' = 2c + \frac{a}{6}(3a^2 + 4a_2)$. These two curves touch in one point and each curve has two points where the vector field (21) is tangent, one of which is this tangency point. We draw the divisor at infinity, together with the three points where the flow is tangent to the divisor, in the following figure.

Figure 2

The two curves are interchanged by the time-involution given by flipping the signs of s_5 and s_7. Since the functions s_1, s_2 and s_3 have a simple pole along each of the hyperelliptic curves at infinity and since the point of tangency of these curves is mapped to $(0 : 0 : 0 : 1)$ by the embedding of the Kummer surface in $\mathbb{P}^3$ by means of the four functions $\{1, s_1, s_2, s_3\}$, we calculate an equation of the Kummer surface in terms of these functions. The result is a quartic equation of the form

$$(s_1 - s_2^2)s_3^2 + P_3(s_1, s_2)s_3 + P_4(s_1, s_2) = 0,$$

for some polynomials f_3, f_4 of degree 3 and 4. Using Theorem 9 we set

$$\begin{aligned} s_2 &= -\frac{1}{2}(\mu_1 + \mu_2), & s_1 &= \mu_1\mu_2 \\ s_5 &= -\frac{1}{8}(\dot{\mu}_1 + \dot{\mu}_2) & s_7 &= -\frac{1}{8}(\dot{\mu}_1\mu_2 + \mu_1\dot{\mu}_2), \end{aligned}$$

Solving the invariants S_1, S_2 and S_5 for s_4, s_6 and s_3 respectively, we are able to express all functions s_i in terms of μ_i and $\dot{\mu}_i$; the two linear equations in $\dot{\mu}_1^2$ and $\dot{\mu}_2^2$ are found by substituting these in the remaining invariants S_3 and S_4. These equations are easily solved as

$$\dot{\mu}_i^2 = \frac{f(\mu_i)}{(\mu_1 - \mu_2)^2},$$

where

$$f(\mu) = (\mu^3 + a_2\mu - 2b_2)^2 - 64(c_2\mu^2 + 2d_2\mu - e_2).$$

The equation $y^2 = f(\mu)$ is seen to be an equation for the curves Γ_ϵ and the Jacobi form for the differential equations corresponding to the vector field (21) follows immediately from it, showing the flow of this vector field is the highest flow with respect to the curve $y^2 = f(\mu)$.

Since there is no physical interpretation for the seven-dimensional system, there is also no natural Poisson structure for this system; there are (at least) three different Poisson structures for the seven-dimensional system. As explained in the previous section, we read off from the equation (25)of the curve that there are three possibilities for J by taking $H = S_1$, $H = S_3$ and $H = S_4$. Taking for example $H = S_3$, the matrix J is given by

$$\begin{pmatrix} 0 & 0 & -8s_7 & 0 & 0 & 0 & -4s_1 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 8s_7 & 0 & 0 & s_7 & -s_6 & -2s_2s_7 & 4s_3 - 2s_2s_6 \\ 0 & 0 & -s_7 & 0 & 2s_2 & 0 & -s_1/2 \\ 0 & 2 & s_6 & -2s_2 & 0 & -s_1/2 & 0 \\ 0 & 0 & 2s_2s_7 & 0 & s_1/2 & 0 & s_1s_2 \\ 4s_1 & 0 & 2s_2s_6 - 4s_3 & s_1/2 & 0 & -s_1s_2 & 0 \end{pmatrix}.$$

The vector field $\dot{X} = J\nabla H_4(X)$ gives a vector field commuting with the vector field (21) and S_1, S_2 and S_5 are Casimir functions. An expression for the symplectic structure can be found by inverting the non-singular minor taken from rows and columns 1, 2, 3 and 5, giving

$$\omega = \frac{ds_1 \wedge ds_3}{8s_7} + \frac{ds_2 \wedge ds_5}{2} - \frac{s_6 ds_1 \wedge ds_2}{16s_7}.$$

We showed that the vector field corresponding to S_3 for this symplectic structure (i.e. the vector field (21)) gave rise to the lowest flow. It is shown in exactly the same way that the vector field corresponding to $2S_4$ gives rise to the highest flow (with respect to the same equation of the curve). An expression for the other two Poisson structures is found easily in the same way and since any pair of brackets are completely bi-Hamiltonian, they are all compatible.

Writing the equation of the curve as $y^2 = f(x) = P^2(\mu) - 64(S_3\mu^2 + 2S_4\mu - e_2)$, it is seen that $2\frac{\partial f}{\partial S_3} = \mu\frac{\partial f}{\partial S_4}$. Applying Proposition 12, ω can be written as $\omega = d\mu_1 \wedge d\nu_1 + d\mu_2 \wedge d\nu_2$ where

$$\sigma_i = \int \frac{d\Delta_i}{\sqrt{P(\mu_i) + 64e_2 + \mu_i\Delta_i}} = \frac{2}{\mu_i}\sqrt{P(\mu_i) + 64e_2 + \mu_i\Delta_i}, \qquad (i = 1, 2),$$

and $\Delta_i = -64(S_3\mu_i + 2S_4)$. Denoting again by $e_1, \ldots, e_4$ four real Weierstrass points on the curve $y^2 = f(\mu)$ the action variable p_1 is given by

$$p_i = 4\int_{e_{2i-1}}^{e_{2i}} \sqrt{P(\mu) + 64e_2 + \mu\Delta}\frac{d\mu}{\mu}.$$

and angle variables follow from differentiation of the generating function with respect to p_1 and p_2 respectively (as in the previous example).

c. A quartic potential

This potential was constructed by Ramani et al. (see [RDG]), as an example of an integrable system which admits only fractional Laurent solutions. It is a system which is a.c.i. in the generalised sense as defined in [AvM1]; see also [Pi2] for an explanation of the nature of the Laurent solutions for such systems. Slightly altered, the Hamiltonian and the extra constant of motion read

$$\begin{aligned} Q_4 &= -4(p_1^2 + 4p_2^2 - \frac{1}{256}(q_1^4 + 3q_1^2q_2^2 + q_2^4)) = a_3, \\ Q_4' &= 4q_1p_1p_2 - q_2p_1^2 - \frac{q_1^2q_2}{256}(2q_1^2 + q_2^2) = b_3, \end{aligned} \tag{26}$$

with the standard symplectic structure. It follows that the vector field corresponding to Q_4 is given by

$$\begin{aligned} \dot{q}_1 &= -8p_1, \qquad & \dot{p}_1 &= -\frac{q_1}{32}(2q_1^2 + 3q_2^2), \\ \dot{q}_2 &= -32p_2, \qquad & \dot{p}_2 &= -\frac{q_2}{32}(3q_1^2 + 2q_2^2), \end{aligned} \tag{27}$$

The system is weight homogeneous with q_1, q_2 having weight 1 and p_1, p_2 weight 2, so that Q_4 and Q_4' have weight 4 and 5 respectively. There are two principal balances, which depend on 3 free parameters. They are special in the sense that they are fractional, i.e., Laurent series in $\sqrt{t}$. The first few terms of q_i for the principal balances are

$$q_1 = \sqrt{\frac{a}{t}}(1 + \frac{at}{4} + \frac{a^2bt^2}{32} + \cdots), \qquad q_2 = \frac{\epsilon}{t}(1 - \frac{at}{2} - \frac{a^2t^2}{4} + \cdots). \tag{28}$$

Here the two balances are distinguished by $\epsilon = \pm 1$. Substituting the series (28) for q_i and p_i into (26) yields two isomorphic curves

$$\Gamma_\epsilon\colon a^5(b^2 + 10b - 39) + 64^2(aa_3 + 4b_3\epsilon) = 0, \qquad (\epsilon = \pm 1), \tag{29}$$

which are smooth hyperelliptic curves of genus two for generic values of a_3 and b_3.

We investigate the affine variety $\mathcal{A}$ given by (26) for generic values of a_3 and b_3. It has a fixed-point free involution τ which maps (q_1, q_2, p_1, p_2) to $(-q_1, q_2, -p_1, p_2)$, hence $\mathcal{A}$ is a double unramified cover of $\mathcal{A}/\tau$. We search for functions in the coordinate ring of $\mathcal{A}/\tau$ which have the property that they behave like $\frac{1}{t}$ or $\frac{1}{t^2}$ at worst when the series (28) are substituted into them. We denote these vector spaces by $\mathcal{L}(\Gamma_1 + \Gamma_{-1})$ and $\mathcal{L}(2\Gamma_1 + 2\Gamma_{-1})$ respectively. For $\mathcal{L}(\Gamma_1 + \Gamma_{-1})$ we find four independent functions $\{1, q_2, q = q_1^2, \tilde{q} = q_1^2 q_2^2 - (16p_1)^2\}$, while $\mathcal{L}(2\Gamma_1 + 2\Gamma_{-1})$ contains the additional twelve independent functions

$$\{q_2^2, p_2, q_1p_1, qq_2, q^2, \{q, q_2\}, \{q, q_2\}q_2 + 16p_1q_1q, \tilde{q}q_2, \tilde{q}q, \{q_2, \tilde{q}\}, \{q, \tilde{q}\}, \tilde{q}^2\}.$$

The sixteen functions $z_0, \ldots, z_{15}$ above induce a map $\varphi\colon \mathcal{A}/\tau \to \mathbb{P}^{15}$ determined by $\varphi(q_1, q_2, p_1, p_2) = (z_0 : \ldots : z_{15})$. Substituting the series (28) in this embedding and letting $t \to 0$ one finds (up to scalar factors) an embedding of the curves Γ_ϵ, in which (a, b) is mapped to

$$(0:0:0:0:1:\epsilon:a:a\epsilon:a^2:a^2\epsilon:a^3b':a^3\epsilon b':a^4b':a^4b'\epsilon:a^5b':a^6b'^2),$$

upon setting $b' = b - 3$. The curves $\varphi(\Gamma_\epsilon)$ are disjoint for finite values of a and b, but intersect in $(0 : \cdots : 0 : 1)$, where these blow up, as in Figure 2.

Applying the methods explained in [AvM1] one shows using the Laurent series and the embedding above that the images $\varphi(\mathcal{A}/\tau), \varphi(\Gamma_1)$ and $\varphi(\Gamma_{-1})$ in fact build up a complete Abelian variety — more precise the Jacobian of Γ_ϵ — lying in projective space. It follows that the system (27) is a.c.i. in the generalised sense. The spaces $\mathcal{L}(\Gamma_1 + \Gamma_{-1})$ and $\mathcal{L}(2\Gamma_1 + 2\Gamma_{-1})$ introduced above can now be understood as meromorphic functions on the Jacobian of Γ_ϵ which have a simple respectively double pole along each of the curves Γ_ϵ only.

Now we want to calculate an affine equation for the Kummer surface of $\mathcal{A}/\tau$. Remark that the intersection point of Γ_1 and Γ_{-1} is mapped to $(0:0:0:1)$ by the map from $\mathrm{Jac}(\Gamma_\epsilon)$ into projective space $\mathbb{P}^3$ by means of $\{1, q_2, q, \tilde{q}\}$. Therefore we use the functions of this base to find the standard equation of the Kummer surface; eliminating p_2 from Q_4 and Q_4' and expressing p_1^2 in terms of q and q_2 (no square roots are needed) a quartic equation

$$(4q^2 + q_2^2)\tilde{q}^2 + P_3(q, q_2)\tilde{q} + P_2^2(q, q_2) = 0,$$

is found (P_2 and P_3 are polynomials of degree 2 and 3 respectively). We are again in position to apply Theorem 9: we put

$$\begin{aligned} q_2 &= \mu_1 = \mu_2, \qquad & q = q_1^2 &= -\mu_1\mu_2, \\ -32p_2 &= \dot\mu_1 + \dot\mu_2, \qquad & 16p_1q_1 &= \dot\mu_1\mu_2 + \mu_1\dot\mu_2. \end{aligned} \tag{30}$$

Since the equations of the invariant manifold depend on q_1p_1 and q_1^2 rather than on p_1 and q_1, these equations can immediately be written as

$$\begin{aligned} &(\mu_1\dot\mu_2^2 - \mu_2\dot\mu_1^2)(\mu_1-\mu_2)^2 + \mu_1\mu_2(\mu_1^5 - \mu_2^5) - 64a_3\mu_1\mu_2(\mu_1-\mu_2) = 0, \\ &(\mu_1^2\dot\mu_2^2 - \mu_2^2\dot\mu_1^2)(\mu_1-\mu_2)^2 + \mu_1^2\mu_2^2(\mu_1^4 - \mu_2^4) - 256b_3\mu_1\mu_2(\mu_1-\mu_2) = 0. \end{aligned}$$

Solving these equations linearly for $\dot\mu_1^2$ and $\dot\mu_2^2$ one finds

$$\dot\mu_i^2 = \frac{\mu_i^6 + 64a_3\mu_i^2 + 256b_3\mu_1}{(\mu_1-\mu_2)^2},$$

leading to the Jacobi form. One proceeds in the same way to find the Jacobi form for the vector field corresponding to Q_4'; the constants appearing in the Jacobi form are $(1/4, 0)$ in this case (and $(0,1)$ for the first vector field). Since the original coordinates q_i and p_i which define the system are Darboux coordinates (the symplectic structure being the standard one), there is no need to construct (other) Darboux coordinates.

d. The master systems

In [M], Mumford constructs a natural vector field on the Jacobian of any hyperelliptic curve of genus g, given by an equation $y^2 = f(x)$, $\deg f = 2g+1$. We adapt his construction here for the case the hyperelliptic curve is given by an equation $y^2 = f(x)$, $\deg f = 2g+2$, and find a new integrable sytem (the *even master system*), whose geometry is totally different from the one constructed by Mumford (the *odd master system*).

We denote by Γ the hyperelliptic curve of genus g defined by $y^2 = f(x)$, $\deg f = 2g+2$, f monic. By a simple translation we may suppose that the coefficient of f in x^{2g+1} vanishes. We denote the two points at infinity by ∞_1 and ∞_2 and remark that they are interchanged by the hyperelliptic involution σ. To each divisor of the form $\mathcal{D} = \sum_{i=1}^g P_i$, with $P_i \in \Gamma \setminus \{\infty_1, \infty_2\}$ and $i \neq j \Rightarrow P_i \neq \sigma P_j$, we associate three polynomials

$$u(x) = \sum_{i=0}^g u_i x^{g-i}, \qquad v(x) = \sum_{i=1}^g v_i x^{g-i}, \qquad w(x) = x^{g+2} + \sum_{i=-1}^g w_i x^{g-i},$$

as follows: $u(x) = \prod_{i=1}^g (x - x(P_i))$ and $v(x)$ is the unique polynomial of degree less than g approximating y to order $\text{mult}_{P_i}(\mathcal{D})$ at P_i, which makes $f(x) - v^2(x)$ divisible by $u(x)$, the quotient being denoted by $w(x)$. Actually, since the coefficient of f in x^{2g+1} was taken to be zero, it follows that $w_{-1} = -u_1$. In order to construct a vector field on Jac(Γ), we consider for a divisor $\mathcal{D} = \sum_{i=1}^{g} P_i$ as above, the function

$$h(x,y) = \frac{u(x)}{y + v(x) + xu(x)},$$

which is (up to a constant) the unique function with zero divisor $\mathcal{D} + \infty_1$. Then $h^{-1}\{0\}$ is $\sum P_i + \infty_1$, and for $|\epsilon|$ small, $h^{-1}\{-\frac{\epsilon}{2}\}$ will be a divisor $\sum P_i^\epsilon + Q^\epsilon$, with P_i^ϵ close to P_i and Q^ϵ close to ∞_1. Now the relation $h(x,y) = -\frac{\epsilon}{2}$ implies $2u(x) + \epsilon v(x) + \epsilon x u(x) = -\epsilon y$, so

$$u(x)\left[u(x) + \epsilon(xu(x) + v(x)) + \frac{\epsilon^2}{4}(x^2 u(x) + 2xv(x) - w(x)\right] = 0.$$

The second factor is a polynomial (in x) of degree $g+1$ and determines the divisor $\sum P_i^\epsilon + Q^\epsilon$. To separate the P_i^ϵ from Q^ϵ for $|\epsilon|$ small, it is only necessary to factorize this polynomial in a factor of degree g and another of degree 1. Looking at the degrees of $u(x), v(x)$ and $w(x)$ the only possible factorisation $(u(x) + \epsilon v(x) + \epsilon^2 \cdots)(1 + \epsilon x + \epsilon^2 \cdots)$ is found. Therefore $\dot{u}(x) = v(x)$ and the time-derivatives of $v(x)$ and $w(x)$ follow immediately from differentiation of $f = u(x)w(x) + v(x)^2$. This gives

$$\begin{aligned}
\dot{u}(x) &= v(x),\\
\dot{v}(x) &= -\frac{1}{2}(w(x) - u(x)(x^2 - 2u_1 x + w_0 + 2u_1^2 - u_2)),\\
\dot{w}(x) &= -v(x)(x^2 - 2u_1 x + w_0 + 2u_1^2 - u_2).
\end{aligned}$$

Moreover g commuting flows can be written down in a compact form as Lax pairs in $sl(2)$:

$$\dot{A} = \frac{1}{2}[A, P_k A + B_k], \qquad A = \begin{pmatrix} v(x) & u(x) \\ w(x) & -v(x) \end{pmatrix}, \tag{31}$$

where $k = 1, \ldots, g$ gives the different flows, P_k is an operator on polynomials

$$P_k(\sum_{i=0}^{g+2} A_i x^i) = \sum_{i=0}^{k+1} A_{g-i+2} x^{1+k-i}, \tag{32}$$

and B_k is a strictly lower triangular matrix whose only non-zero entry equals $b_k = -u_k x + 2u_1 u_k - u_{k+1}$. The case of interest in this paper is the genus two case. Five invariants are found from the relation $f = u(x)w(x) + v(x)^2$,

$$\begin{aligned}
V_1 &= w_0 - u_1^2 + u_2 = a_4,\\
V_2 &= w_1 + w_0 u_1 - u_1 u_2 = b_4,\\
V_3 &= w_2 + w_1 u_1 + u_2 w_0 + v_1^2 = c_4,\\
V_4 &= w_1 u_2 + w_2 u_1 + 2v_1 v_2 = d_4,\\
V_5 &= u_2 w_2 + v_2^2 = e_4,
\end{aligned} \tag{33}$$

when $f(x)$ is written as $f(x) = x^6 + a_2x^4 + b_2x^3 + c_2x^2 + d_2x + e_2$. Before searching the Laurent solutions we note the system is weight homogeneous with weight i for u_i, $i+1$ for v_i and $i+2$ for w_i. Then V_i has weight $i+1$. There are two principal balances, depending on six free parameters. We only display the principal balances for the u variables since they determine the solutions for the v and w variables upon using the differential equations:

$$\begin{aligned} u_1 &= \pm\frac{1}{t}(1 + at + bt^2 + ct^3 + et^4 + ft^5 + \cdots), \\ u_2 &= \frac{1}{t}(2a - 2a^2t + dt^2 + (2ac - ad + 2a^2b)t^3 + \cdots. \end{aligned} \tag{34}$$

Substituting the series (34) into the invariants (33) gives two copies of Γ, say Γ_1 and Γ_2, to be adjoined to the invariant affine surface yielding an Abelian variety. To show this, one searches for functions which behave like t^{-1} and those which behave like t^{-2}. Using the series (33), a base for $\mathcal{L}(\Gamma_1 + \Gamma_2)$ is found, $\{1, u_1, u_2, u = w_2 + u_1w_1 + u_1^2w_0\}$, and $\mathcal{L}(2\Gamma_1 + 2\Gamma_2)$ contains 12 additional independent functions,

$$\{v_1, v_2, u_1^2, u_1u_2, u_1u, u_2^2, u_2u, u^2, \dot{u}, \{u_1, u_2\}, \{u_1, u\}, \{u_2, u\}\},$$

which embeds the invariant affine surface in projective space. Using the series (34), the image is compactified by adding two translates of Γ which touch at one point, as in Figure 2. Expressing the invariants (33) in terms of the base $\{1, u_1, u_2, u\}$, a quartic equation

$$(u_1^2 - 4u_2)u^2 + f_3(u_1, u_2)u + f_4(u_1, u_2) = 0,$$

for the Kummer surface is obtained (f_3 and f_4 have degree 3 and 4 respectively). By Proposition 9 we can linearise both vector fields by setting $u_1 = -\mu_1 - \mu_2$, and $u_2 = \mu_1\mu_2$.

Finally we take a look at the symplectic structures of the even master system. For $i = 0, \ldots, 3$, any of the three pairs (V_{i+1}, V_{i+2}) of constants of motion can be taken as Hamiltonians defining the two commuting vector fields described by the Lax pair (for $k = 1, 2$) with respect to some symplectic structure. For example, take

$$J = \frac{1}{2}\begin{pmatrix} 0 & -U & 0 & 2V \\ U & 0 & X & -W \\ 0 & -{}^tX & 0 & -{}^tY \\ -2V & W & Y & 0 \end{pmatrix},$$

with

$$U = \begin{pmatrix} u_1 & u_2 \\ u_2 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 2u_1^2 - u_2 \\ 2u_1u_2 \end{pmatrix}, \quad Y = 2\begin{pmatrix} v_2 - 2u_1v_1 \\ -2u_1v_2 \end{pmatrix},$$

V and W being constructed in exactly the same way as U. Then $\dot{X} = J\nabla V_2$ and $\dot{X} = J\nabla V_3$ are exactly the two commuting vector fields (31) for $k = 1, 2$. The left upper 4×4 minor of J is non-singular and its inverse gives an expression

$$\omega_1 = 2(\frac{du_1 \wedge dv_2}{u_2} + \frac{du_2 \wedge dv_1}{u_2} - \frac{u_1 du_2 \wedge dv_2}{u_2^2})$$

for the symplectic two-form corresponding to this choice of Hamiltonians. In the same way the other three symplectic structures ω_0, ω_2 and ω_3 are found to be

$$\begin{aligned}
\omega_0 &= 2\Big(\frac{du_1 \wedge dv_1}{u_2} - \frac{u_1 du_1 \wedge dv_2}{u_2^2} - \frac{u_1 du_2 \wedge dv_1}{u_2^3} - \frac{u_2 - u_1^2}{u_2^3} du_2 \wedge dv_2\Big),\\
\omega_2 &= 2(-du_1 \wedge dv_1) + \frac{du_2 \wedge dv_2}{u_2},\\
\omega_3 &= 2(u_1 du_1 \wedge dv_1) - du_1 \wedge dv_2 - du_2 \wedge dv_1.
\end{aligned}$$

Let $(F_1, F_2) = (V_{i+1}, V_{i+2})$ for $i = 0, 1, 2$ or 3. Then the polynomial $f(\mu)$ defining Γ satisfies $\frac{\partial f}{\partial F_1}(\mu) = \mu \frac{\partial f}{\partial F_2}(\mu)$, hence setting $\Delta_j = \mu_j F_1 + F_2$, any of these ω_i can be written as $\omega_i = d\mu_1 \wedge d\nu_1 + d\mu_2 \wedge d\nu_2$ where

$$\nu_j = \int \frac{d\Delta_j}{\sqrt{f(\mu_j)}} = 2\frac{\sqrt{f(\mu_j)}}{\mu_j^{3-i}}, \qquad (j = 1, 2).$$

As before action-angle coordinates derive immediately from it.

We shortly describe now the odd master system (see [M]). In the same way as before one associates three polynomials u, v and w to divisors on an algebraic curve, the degree of w being $g+1$ in this case (instead of $g+2$). Its Lax pair takes the same form as in (31) with $b_k = -u_k$, and the invariants are again found from the relation $f(x) = u(x)w(x) + v^2(x)$. In the genus two case, the complete variety is the union of the affine piece in which we have the coordinates u_i, v_i, w_i and one copy of the curve Γ. We draw the divisor at infinity as follows.

Figure 3

The roots of $u(x)$ are linearising variables, exactly as in the even master system.

It is easy to check that this system has 4 independent symplectic structures, like the even master system. All computations concerning the construction of Darboux coordinates for each of this symplectic structures as well as the calculation of the corresponding action-angle variables give exactly the same formulas as for this system, so we do repeat these formulas here.

e. Integrable Hénon-Heiles

The Hénon-Heiles system is another example of a potential problem, but this well known potential is of the third degree. The Hamiltonian and an extra invariant are given by

$$\begin{aligned}
Q_3 &= 2(p_1^2 + 4p_2^2) - \frac{q_2}{8}(2q_1^2 + q_2^2) = a_6,\\
Q_3' &= q_2 p_1^2 - 4q_1 p_1 p_2 + \frac{q_1^2}{16}(q_1^2 + q_2^2) = b_6,
\end{aligned} \tag{35}$$

and the symplectic structure is again the standard one, so that two commuting vector fields are given by

$$\begin{aligned} \dot{q}_1 &= 4p_1, & \dot{p}_1 &= \frac{q_1 q_2}{2}, \\ \dot{q}_2 &= 16p_2, & \dot{p}_2 &= \frac{2q_1^2 + 3q_2^2}{8}. \end{aligned}$$

The system is weight homogeneous, with weight 2 for q_i and weight 3 for p_i. Then Q_3 and Q_3' have weight 6 and 8 respectively. There is one principal balance depending on three free parameters; the first few terms for q_i are given by

$$q_1 = \frac{1}{t}(a + \frac{a^3t^2}{3} + bt^3 - \frac{2}{9}a^5t^4 - \frac{1}{3}a^2bt^5 + \cdots), \qquad q_2 = \frac{1}{t^2}(1 - \frac{a^2t^2}{3} - \frac{a^4t^4}{3} - \frac{4}{3}abt^5 + ct^6 + \cdots),$$

the series for p_i are found by differentiation. The eight functions with at worst a double pole at infinity $\{1, q_1, q_2, p_1, q_1^2, q_2p_1 - 2q_1p_2, q_1^2q_2 - 16p_1^2, q_1q_2^2 - 32p_1p_2\}$, give (for generic values of a_6 and b_6) an embedding of the affine variety defined by the invariants (35) in $\mathbb{P}^7$. Using this embedding one sees that the affine variety is compactified by adjoining a genus three curve $\mathcal{C}$, which induces a polarisation of type $(1,2)$ on the surface. An equation for $\mathcal{C}$ is found by substituting these series into the invariants Q_3 and Q_3',

$$\mathcal{C}\colon b^2 = b_6 + \frac{1}{2}a_6a^2 - \frac{1}{16}a^8.$$

We draw this hyperelliptic curve of genus three as follows.

Figure 4

As an example of the linearisation procedure for Abelian surfaces which do not carry a principal polarisation, we now show how to linearise the Hénon-Heiles system. We want to stress that although the linearisation is done formally in a very similar way as for the quartic potential, the geometry of both systems is totally different: for the Hénon-Heiles potential the affine invariant surfaces complete into Abelian surfaces, while for the quartic potential they don't. Letting Γ be the genus two curve

$$y^2 = b_6 + \frac{1}{2}a_6x - \frac{1}{16}x^5,$$

the curve $\mathcal{C}$ is easily seen to be a two-fold unramified cover of Γ, the map being given by $x = a^2$, $y = ab$. The involution $(a,b) \to (-a,-b)$ flipping the sheets of the cover can be extended to an involution τ on the invariant manifold $\mathcal{T}^2$ by $(q_1, q_2, p_1, p_2) \to (-q_1, q_2, -p_1, p_2)$. Since this involution preserves the vector field, it amounts to translation by a half period and the quotient $\mathcal{T}^2/\tau$ is an Abelian surface containing the smooth genus two curve Γ, hence $\mathcal{T}^2/\tau = \mathrm{Jac}(\Gamma)$. Using the base in (35) it is easy to see that the functions

$\{1,\, \theta_1 = q_2,\, \theta_2 = -q_1^2,\, \theta_3 = q_1^2 q_2 - 16p_1^2\}$ are invariant under τ, hence go down to $\mathrm{Jac}(\Gamma)$. Using the invariants (35) the Kummer surface of $\mathrm{Jac}(\Gamma)$ is found to be

$$(\theta_1^2 - 4\theta_2)\theta_3^2 + f_3(\theta_1, \theta_2) + f_4(\theta_1, \theta_2) = 0.$$

By Theorem 9, adapted to the case where the invariant tori do not carry a principal polarisation, we let

$$\begin{aligned} \theta_1 &= q_2 = -\mu_1 - \mu_2, & \dot\theta_1 &= \dot q_2 = 16p_2 = -\dot\mu_1 - \dot\mu_2, \\ \theta_2 &= -q_1^2 = \mu_1\mu_2, & \dot\theta_2 &= -2q_1\dot q_1 = -8q_1p_1 = \dot\mu_1\mu_2 + \mu_1\dot\mu_2, \end{aligned}$$

and write the invariants (35) in terms of μ_1, μ_2 and their derivatives (no square roots are needed). The resulting two polynomials are solved linearly for $\dot\mu_i^2$ as

$$\dot\mu_i^2 = \frac{\mu_i^5 + 8a_6\mu_i^2 - 16b_6\mu_i}{(\mu_1 - \mu_2)^2},$$

leading immediately to the Jacobi form. For the vector field corresponding to Q_3' the computation is very similar.

Let π denote the natural projection $\pi\colon \mathcal{T}^2 \to \mathcal{T}^2/\tau$. We calculate the divisor $\pi^*\Delta$ on $\mathcal{T}^2$. Setting $q_2 = 2\epsilon i\ (\epsilon = \pm 1)$ two curves Δ_+ and Δ_- are found on the invariant surface. They are given by an equation

$$64p_1^2 q_1(8a_6q_1 - \epsilon i q_1^4 + 16b_6 i\epsilon) - (16b_6 + 3q_1^4)^2 = 0, \tag{36}$$

and they are both isomorphic to Γ. Since the generic curve in the pencil has virtual genus $\dim \mathcal{L}(2\Gamma) + 1 = 9$, the embedded curves Δ_+ and Δ_- must have singular points. To find them we substitute $q_2 = 2i\epsilon q_1$ in the embedding (35) in which p_2 is eliminated using the invariant Q_3'. Then it is clear that there can only be singular points at infinity, i.e. where q_1 or q_2 blow up. We get three cases

1) $q_1 = q_\delta + u^2$, $p_1 = \frac{1}{u}(c + du^2)$, where q_δ is any of the four roots of $-\epsilon i q^4 + 8a_6 q + 16b_6 i\epsilon$. Then $u \mapsto (0 : \cdots : 1 : \epsilon i) + \frac{u}{c}(0 : 0 : 0 : 1 : 0 : \epsilon i q_\delta : 0 : 0)$, which means that each of the curves has a four-fold point (at least).
2) $q_1 = \frac{4i}{u^2}$, $p_1 = \frac{3}{u^3}(1 + cu^6)$ or $\frac{3i}{u^3}(1 + cu^6)$ according to $\epsilon = 1, -1$. Then $u \mapsto (0 : \cdots : 1 : -\epsilon i) + \frac{u}{4}(0 : \cdots : 0 : 1 : 0 : 0)$, which means that each curve goes through the four-fold singularity of the other curve.
3) $q_1 = u^2$, $p_1 = \frac{c}{u}(1 + du^2)$ where $b_6 = 4i\epsilon c^2$. Then $u \mapsto (0 : \cdots : 1 : -\epsilon i) - \frac{u}{16c}(0 : 0 : 0 : 1 : 0 : \cdots : 0)$, which means that each curve has a double point where the other one has his four-fold point.

Now an ordinary four-fold point accounts for a drop in genus of at least $\binom{4}{2} = 6$ while a double point accounts for at least $\binom{2}{2} = 1$, with equality for ordinary points only. Therefore $2 = g(\Gamma) \le 9 - 6 - 1 = 2$, and it follows that all inequalities are in fact equalities. Consequently each curve has an ordinary four-fold point and an ordinary double point and no other singularities. Also both curves intersect at least in their singular points, giving

2.(2.4) = 16 intersection points at least, with equality for normal intersection only. By (4) it follows that $\frac{\Delta_+.\Delta_-}{2} = g(\Delta_+) - 1 = 8$, so that Δ_+ and Δ_- have their singular points as their only intersection points and have distinct tangents at these points. The two singular points are half periods and are the intersection points of the two curves with the curve at infinity. Remark that the divisor has two (ordinary) six-fold points (and no other singularities) as was to be expected since the divisor is an unramified cover of a curve with one (ordinary) six-fold point (only). Also it is easy to see that these two points belong to the curve $\mathcal{C}$, since the six-fold point of Δ belongs to $\pi(\mathcal{C}) = \Gamma$.

The singularities of the curves can be seen from the following picture.

Figure 5

For simplicity only one curve is drawn, the other curve can be drawn by a reflection of the first curve which exchanges the two singular points.

As for the construction of Darboux coordinates for the Hénon-Heiles system, the same remark we made in the case of the quartic potential applies since again the original variables defining the system are Darboux coordinates.

f. Kowalevski's top

The best known system in the list is undoubtly Kowalevski's top. We refer to [AvM2], [BRS], [HH] and [HvM] for an extensive discussion of the geometry of this top and the construction of a Lax pairs for this top.

Recall that the equations describing the motion of this top are

$$\begin{aligned} \dot{k}_1 &= k_2k_3, & \dot{l}_1 &= 2k_3l_2 - k_2l_3, \\ \dot{k}_2 &= 2l_3 - k_1k_3, & \dot{l}_2 &= k_1l_3 - 2k_3l_1, \\ \dot{k}_3 &= -2l_2, & \dot{l}_3 &= k_2l_1 - k_1l_2, \end{aligned}$$

with constants of motion

$$\begin{aligned} K_1 &= \frac{1}{2}(k_1^2 + k_2^2) + k_3^2 + 2l_1 = a_7, \\ K_2 &= k_1l_1 + k_2l_2 + k_3l_3 = b_7, \\ K_3 &= l_1^2 + l_2^2 + l_3^2 = c_7, \\ K_4 &= [\frac{1}{4}(k_1 + ik_2)^2 - (l_1 + il_2)][\frac{1}{4}(k_1 - ik_2)^2 - (l_1 - il_2)] = d_7. \end{aligned} \tag{37}$$

The vector field is weight homogeneous if we give k_i weight 1 and l_i weight 2; the invariants have weight 2, 3, 4, 4 respectively. There are two principal balances with five free parameters. Using the functions $x_{1,2} = \frac{k_1 \pm ik_2}{2}$, $y_{1,2} = (\frac{k_1 \pm ik_2}{2})^2 - (l_1 \pm il_2)$, $x_3 = k_3$, $y_3 = l_3$ as new coordinates, one principal balance is given by

$$x_1 = \frac{a}{t} + b(1+a^2) + \frac{a}{2}(c + a^2b^2)t + \cdots, \qquad y_1 = \frac{1}{t^2}(1+a^2) + \frac{2ab}{t}(1+a^2) + \cdots,$$

$$x_2 = b - ab^2t + \cdots, \qquad y_2 = \frac{t^2}{a^2}((a^2+1)b^4 - a_7b^2 + 2b_7b + \cdots,$$

$$x_3 = -\frac{1}{t} + ab + ct + \cdots, \qquad y_3 = -\frac{b}{t} + ab^2 + \cdots,$$

the other one is given by interchanging x_1 with x_2, y_1 with y_2 and changing the signs of x_3 and y_3. The 8 independent functions $\{1, x_1, x_2, x_3, y_3, x_1x_2, y_3(x_1+x_2) - x_1x_2x_3, \{x_3, y_3\}\}$ behave like $\frac{1}{t}$ at worst when any of the principal balances is substituted in them. It is proved in [AvM1] that the closure of the image of the corresponding map of the affine invariant surface into projective space $\mathbb{P}^7$ is an Abelian surface, which receives a polarisation of type $(2,4)$ by the two genus three curves $\mathcal{E}_1$ and $\mathcal{E}_2$ which have to be adjoined to the image to get the complete variety. An equation for these curves is given by

$$(a^2+1)^2b^4 - (a^2+1)(a_7b^2 - 2b_7b + c_7) + d_7 = 0.$$

Using the embedding (37) it is easy to check that the two genus three curves intersect (transversally) in four points. These points are half periods and each curve passes through four extra half periods, as in the following figure (the half periods are represented by dots).

Figure 6

These two curves of genus 3 are uninteresting for finding the linearisation since they aren't unramified covers of genus 2 curves. One way to proceed would be to search in the linear system $|\mathcal{E}_1|$ (or $|\mathcal{E}_2|$) for genus 3 curves which cover a hyperelliptic curve (there are actually six of them; see [HvM]). Alternatively we will search in the linear system $|\mathcal{E}_1 + \mathcal{E}_2|$ for a suitable curve. Since each smooth curve in $|\mathcal{E}_1 + \mathcal{E}_2|$ has genus 9, we look for a curve of genus 9 which is an eight-fold cover of a genus 2 curve. By a theorem of Barth (see [Ba]), any affine part of $\mathcal{T}^2$ obtained by removing the zero locus of an odd theta function in $H^0(\mathcal{T}^2, \theta[\mathcal{E}_1 + \mathcal{E}_2])$ can be defined by four quadratic equations involving the six odd Abelian functions with a simple pole along this zero locus. Since the (-1)-eigenspace for the involution $\sigma_0\colon (x_1, x_2, x_3, y_1, y_2, y_3) \to (x_2, x_1, x_3, y_2, y_1, y_3)$, has $\{x_1 - x_2, \{x_3, y_3\}\}$ as a base, we look in the pencil of curves $\lambda(x_1 - x_2) + \mu(x_3\dot{y}_3 - \dot{x}_3y_3) = 0$ and find one particularly simple curve by taking $\mu = 0$, i.e., $x_1 - x_2 = 0$, or, equivalently, $k_2 = 0$. This curve $\mathcal{C}$ is an eight-fold cover of the hyperelliptic genus two curve (*Kowalevski's curve*)

$$\Gamma\colon y^2 = (16x^3 + 4a_7x^2 + 4(c_7 - d_7)x + a_7c_7 - b_7^2 - a_7d_7)(x^2 - \frac{d_7}{4}),$$

and by the theorem of Barth above, the affine surface $\mathcal{T}^2 \setminus \mathcal{C}$ can be described by four quadratic equations in terms of the 6 odd functions (under σ_0) in

$$\mathcal{L}(\mathcal{C}) = \{1, z_1 = \frac{i}{x_1 - x_2}, z_2 = i\frac{x_1 + x_2}{x_1 - x_2}, z_3 = i\frac{x_1 x_2}{x_1 - x_2}, z_4 = i\frac{x_3}{x_1 - x_2}, z_5 = i\frac{y_3}{x_1 - x_2}, \\ z_6 = i\frac{x_1 x_2 x_3 - (x_1 + x_2) y_3}{x_1 - x_2}, z_7 = \frac{x_3 \dot{y}_3 - \dot{x}_3 y_3}{x_1 - x_2}\},$$

i.e., in terms of $z_1, \ldots, z_6$. Using the definitions of these functions above and the original invariants (37), these quadratic equations are found to be (see [HH])

$$\begin{aligned}
&4 z_1 z_3 - z_2^2 = 1, \\
&z_3 z_4 - z_1 z_6 - z_2 z_5 = 0, \\
&(c_7 - d_7) z_1^2 - z_3^2 - b_7 z_1 z_2 + a_7 z_1 z_3 - z_5^2 - z_4 z_6 = 0, \\
&b_7^2 z_1^2 + a_7 (z_3^2 + z_5^2) - z_6^2 + (c_7 - d_7)(z_2^2 + z_4^2 - z_1^2) - 2 b_7 (z_2 z_3 + z_4 z_5) = d_7.
\end{aligned}$$

The next step is to find the covering transformations of the cover $\mathcal{T}^2 \to \mathrm{Jac}(\Gamma)$. Since it was shown by [HvM] that the monodromy group of the cover is $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$ we search for three independent involutions on $\mathcal{T}^2$. Since involutions are given in well-chosen coordinates by flipping some of the signs of the variables, we will put the invariants in normal form. It is easy to reduce the first equation to a sum of squares, $\langle p, p \rangle = 1$, where $p = (p_1, p_2, p_3)$ by setting $p_1 = z_1 + z_3$, $p_2 = -i z_2$, $p_3 = i(z_3 - z_1)$. Then the second quadratic equation reduces to $p_1 (z_4 - z_6) - i p_3 (z_4 + z_6) - 2 i p_2 z_5 = 0$. Therefore we set $l_1 = \frac{z_4 - z_6}{2}$, $l_2 = -i z_5$, $l_3 = \frac{z_4 + z_6}{2i}$. Then the other invariants are given by

$$\begin{aligned}
&\langle l, l \rangle + \langle Qp, p \rangle = -\frac{a_7}{4}, \\
&4 \langle Ql, l \rangle - 4 \det Q \langle Q^{-1} p, p \rangle = d_7,
\end{aligned}$$

with $l = (l_1, l_2, l_3)$ and

$$Q = \frac{1}{4} \begin{pmatrix} c_7 - d_7 - 1 & -i b_7 & i(1 + c_7 - d_7) \\ -i b_7 & -a_7 & b_7 \\ i(1 + c_7 - d_7) & b_7 & 1 - c_7 + d_7 \end{pmatrix}.$$

If Q would have diagonal form, 3 involutions would be given immediately by changing the sign of any pair p_i, q_i and leaving the other variables fixed. Therefore we diagonalise the matrix Q, $\tilde{Q} = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3) = R Q R^{-1}$ and set $q = Rp$, $m = Rl$. Now the affine surface is given by

$$\begin{aligned}
&q_1^2 + q_2^2 + q_3^2 = 1, \\
&q_1 m_1 + q_2 m_2 + q_3 m_3 = 0, \\
&m_1^2 + m_2^2 + m_3^2 + \lambda_1 q_1^2 + \lambda_2 q_2^2 + \lambda_3 q_3^2 = -\frac{a_7}{4}, \\
&\lambda_1 m_1^2 + \lambda_2 m_2^2 + \lambda_3 m_3^2 - (\lambda_2 \lambda_3 q_1^2 + \lambda_1 \lambda_3 q_2^2 + \lambda_1 \lambda_2 q_3^2) = \frac{d_7}{4},
\end{aligned} \tag{38}$$

and the three involutions are given by changing the sign of q_i and m_i ($i = 1, 2$ or 3). The functions $1, q_1^2, q_2^2, m_1^2$ are independent functions which go down to $\mathrm{Jac}(\Gamma)$ (the functions $1, q_1^2, q_2^2$ and q_3^2 are not independent since $q_1^2 + q_2^2 + q_3^2 = 1$). We get an equation for the Kummer surface of $\mathrm{Jac}(\Gamma)$ of the form

$$[\{(\lambda_2+\lambda_3)r_1+(\lambda_3+\lambda_1)r_2+(\lambda_1+\lambda_2)r_3\}^2-4(\lambda_2\lambda_3r_1+\lambda_3\lambda_1r_2+\lambda_1\lambda_2r_3)]m^2 \\ + f_3(r_1,r_2,r_3)m + f_4(r_1,r_2,r_3) = 0,$$

where we used r_3 as an abbreviation for $1 - r_1 - r_2$ and $r_1 = q_1^2$, $r_2 = q_2^2$, $m = m_1^2$. Using Theorem 9 we get the transformation

$$\begin{aligned}(\lambda_2+\lambda_3)r_1+(\lambda_3+\lambda_1)r_2+(\lambda_1+\lambda_2)r_3 &= -\mu_1-\mu_2,\\ \lambda_2\lambda_3r_1+\lambda_3\lambda_1r_2+\lambda_1\lambda_2r_3 &= \mu_1\mu_2.\end{aligned}$$

Taking the derivative of these equations, the invariants are written in terms of $\mu_i, \dot\mu_i$ as

$$\begin{aligned}(\mu_1-\mu_2)\frac{\dot\mu_1^2}{F(\mu_1)}-(\mu_1-\mu_2)\frac{\dot\mu_2^2}{F(\mu_2)} &= -\mu_1-\mu_2\\ (\mu_1-\mu_2)\mu_2\frac{\dot\mu_1^2}{F(\mu_1)}-(\mu_1-\mu_2)\mu_1\frac{\dot\mu_2^2}{F(\mu_2)} &= \mu_1\mu_2+\frac{d_7}{4},\end{aligned}$$

the polynomial F being defined by $F(\mu) = (\mu+\lambda_1)(\mu+\lambda_2)(\mu+\lambda_3)$. Solving for $\dot\mu_i^2$ one gets

$$\frac{\dot\mu_i^2}{F(\mu_i)} = \frac{-\mu_i^2+d_7/4}{(\mu_1-\mu_2)^2},$$

which puts the differential equations in the Jacobi form. We want to remark that

$$\begin{aligned}F(\mu) &= \mu^3+\sum\lambda_i\mu^2+(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1)\mu+\lambda_1\lambda_2\lambda_3\\ &= \mu^3-\operatorname{Tr}Q\mu^2+\frac{(\operatorname{Tr}Q)^2-\operatorname{Tr}Q^2}{2}\mu+\det Q,\\ &= \mu^3+\frac{a_7}{4}\mu^2+\frac{c_7-d_7}{4}\mu+\frac{b_7^2-a_7c_7+a_7d_7}{16},\end{aligned}$$

since

$$\begin{aligned}\sum\lambda_i &= \operatorname{Tr}Q = -\frac{a_7}{4},\\ \lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1 &= \frac{(\operatorname{Tr}Q)^2-\operatorname{Tr}Q^2}{2} = \frac{c_7-d_7}{4},\\ \lambda_1\lambda_2\lambda_3 &= \det Q = \frac{b_7^2-a_7c_7+a_7d_7}{16}.\end{aligned}$$

From this expression for $F(\mu)$ one sees that our curve is isomorphic to the one used by Kowalevski.

We now turn our attention to the symplectic structure and show how to find Darboux coordinates for the Kowalevski top — then action-angle variables can be deduced from it as in the previous examples (see [VN]). Taking the gradient of K_1, it is easy to find the matrix which defines the symplectic structure; it can be written in a compact form as

$$\begin{pmatrix} K & L \\ L & 0 \end{pmatrix}, \text{ where } K = \begin{pmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{pmatrix} \text{ and } L = \begin{pmatrix} 0 & -l_3 & l_2 \\ l_3 & 0 & -l_1 \\ -l_2 & l_1 & 0 \end{pmatrix}.$$

This shows that K_2 and K_3 are Casimir functions, whereas using K_4 it is only a matter of calculation to find a flow commuting with the above one and to check that the constants appearing in the Jacobi form for these differential equations (using the same equation for the curve) are exactly $(-2,0)$ (for a short argument, see [F]) so that we are in position to apply Proposition 12 again. Letting F_1 and F_2 denote the invariants $2K_1$ and $-K_4$ respectively the equation of Kowalevski's curve, as it appears in the linearisation, is given by

$$y^2 = f(\mu) = (-\mu^3 + \frac{F_1}{8}\mu^2 - \frac{c_7 + F_2}{4}\mu + \frac{F_1 c_7 - 2b_7^2 + F_1 F_2}{32})(\mu^2 + \frac{F_2}{4}).$$

By differentiation it is checked immediately that

$$\mu\frac{\partial f}{\partial F_2}(\mu) - \frac{\partial f}{\partial F_1}(\mu) + \phi\frac{\partial f}{\partial \mu}(\mu) = 0.$$

for $\phi = -1/8$. By Proposition 12 it follows that the symplectic structure is given by

$$\omega = \sum_{i=1}^{2}\left[\frac{\mu_i d\mu_i \wedge dF_1}{\sqrt{f(\mu_i)}} + \frac{d\mu_i \wedge dF_2}{\sqrt{f(\mu_i)}} - \frac{dF_1 \wedge dF_2}{8\sqrt{f(\mu_i)}}\right].$$

By Proposition 13 we find Darboux coordinates ρ_1, ρ_2, σ_1 and σ_2 for the Kowalevski top; letting $\Delta_i = F_1\rho_i + F_2 + \frac{F_1^2}{16}$, $E_i = \rho_i^2 + \frac{\Delta_i}{4}$ and $F_i = -\frac{\rho_i c_7}{4} - \frac{b_7^2}{32}$, Darboux coordinates are given by

$$\rho_i = \mu_i - \frac{F_1}{8} \qquad \sigma_i = \int \frac{dE_i}{\sqrt{E_i(F_i - \rho_i E_i)}}$$

the latter integral being easily expressed in terms of trigoniometric functions (or equivalently logarithms) by the rules of calculus.

To finish this more elaborate example we remark that the inverse image of the very singular curve Δ on the invariant surface consists of eight singular divisors of virtual genus three; they come in two groups of four divisors, one group is linearly equivalent to $\mathcal{E}_1$ while the other is linearly equivalent to $\mathcal{E}_2$. Each group passes simply through four fixed half periods and has a ordinary double point in one of the half periods through which each curve of the other group passes (simply), hence the divisor has eight (ordinary) six-fold points and no other singularities since the divisor is an eight-fold unramified cover of a curve with a six-fold point. Therefore the curves which desingularise these diviors all have genus two. These curves were first found by Horozov and van Moerbeke (see [HvM]) when searching for singular curves linearly equivalent to $\mathcal{E}_1$ and $\mathcal{E}_2$. For simplicity, we only draw one curve of each group; the six other curves are found by succesive rotation of the picture over 90 degrees.

Figure 7

7. Transformations between a.c.i. systems

We have studied five examples of a.c.i. systems whose invariant tori carry a principal polarisation (strictly speaking, the quartic potential is only a.c.i. in the generalised sense). We want to show how these are related. To do this we show that we can define a (rational) map from every two-dimensional system whose invariant tori are principally polarised to either the even or the odd master system. Actually, fixing any two vector fields on both systems, there is a unique map which preserves these vector fields and which we can construct explicitely. Since both master systems are defined by Lax pairs in *sl*(2), we get as a by-product a (family of) Lax pair(s) in *sl*(2) for all principally polarised two-dimensional systems. The map is given by the following proposition:

Proposition 15 *Let F_1 and F_2 be any two independent Hamiltonians for an a.c.i. system whose generic invariant tori carry a principal polarisation. Then there is a unique map from this system to either the even or the odd master system which maps the Hamilton vector fields induced by F_1 and F_2 to (multiples of) the weight homogeneous vector fields (31) defining the master systems. The master systems themselves are related by a homographic transformation on the curve Γ underlying the system.*

Proof

Since F_1 and F_2 are independent, we know from Lemma 11 that for some unique equation $y^2 = f(\mu)$ of the curve, the Jacobi form for the differential equations is given by

$$\frac{X_{F_1}\mu_1}{\sqrt{f(\mu_1)}} + \frac{X_{F_1}\mu_2}{\sqrt{f(\mu_2)}} = 0, \qquad \frac{X_{F_2}\mu_1}{\sqrt{f(\mu_1)}} + \frac{X_{F_2}\mu_2}{\sqrt{f(\mu_2)}} = 1,$$
$$\frac{\mu_1 X_{F_1}\mu_1}{\sqrt{f(\mu_1)}} + \frac{\mu_2 X_{F_1}\mu_2}{\sqrt{f(\mu_2)}} = 1, \qquad \frac{\mu_1 X_{F_2}\mu_1}{\sqrt{f(\mu_1)}} + \frac{\mu_2 X_{F_2}\mu_2}{\sqrt{f(\mu_2)}} = 0.$$

By scaling F_1 and F_2 by a common factor (if necessary) we may suppose f to be monic. It follows immediately that

$$\sqrt{f(\mu_i)} = (-1)^{i-1}(\mu_1 - \mu_2) X_{F_1}\mu_i.$$

Let $u(x) = (x - \mu_1)(x - \mu_2)$ and $v(x) = X_{F_1}u(x)$ then

$$\begin{aligned} v(\mu_i) &= X_{F_1}[(x-\mu_1)(x-\mu_2)]_{x=\mu_i}, \\ &= (-1)^i(\mu_1 - \mu_2)X_{F_1}\mu_i, \\ &= -\sqrt{f(\mu_i)}, \end{aligned}$$

which shows that $f(x) - v^2(x)$ is divisible by $u(x)$ and we can define $w(x)$ to be the quotient. Depending on the degree of $f(x)$ this defines the unique mapping to either the even or the odd master system. To show that the mapping is rational, recall that the coefficients u_1 and u_2 of $u(x)$ are meromorphic functions when restricted to the (generic) invariant tori, therefore they are rational functions in the original phase variables. Since the differential equations are rational (or polynomial) in the original variables, $v(x)$ is also rational and the rationality of $w(x)$ follows immediately. Remark that by construction the vector fields generated by F_1 and F_2 are mapped to the natural vector fields on the even (or odd) master system indeed.

In order to relate the odd and even master system, recall that in both cases $u(x) = x^2+u_1x+u_2$ is defined as $u(x) = (x-x(P_1))(x-x(P_2))$ with respect to some equation of the curve and $v(x) = \dot{u}(x)$. If two equations $y^2 = f(x)$, $\deg f = 5$ and $y'^2 = f'(x')$, $\deg f = 6$ are given then x and x' are related by a homographic transformation, $x' = \frac{\alpha x+\beta}{\gamma x+\delta}$ which sends a Weierstrass point to infinity. Letting $u(x) = x^2 + u_1x + u_2$ and $u'(x') = x'^2 + u_1'x' + u_2'$ one finds by direct calculation

$$\begin{aligned} u_1' &= -x'(P_1) - x'(P_2), \\ &= \frac{(\alpha\delta+\beta\gamma)u_1 - 2\alpha\gamma u_2 + 2\beta\delta}{\gamma^2 u_2 - \gamma\delta u_1 + \delta^2}, \\ u_2' &= x'(P_1)x'(P_2), \\ &= \frac{\alpha^2 u_2 - \alpha\beta u_1 + \beta^2}{\gamma^2 u_2 - \gamma\delta u_1 + \delta^2}, \end{aligned}$$

and the corresponding polynomial $v(x')$ is expressed in terms of $u(x), v(x)$ by differentiation. Also $w'(x')$ follows from the curve relation $u'(x')w'(x') + v'^2(x') = f'(x')$. ∎

It follows from the proof of the theorem that the map is easy to construct once the linearising variables are found (which can be done by the methods explained in Section 3). This is illustrated in the following examples. At first since the Toda lattice linearises by setting

$$\begin{aligned} t_6 &= \mu_1 + \mu_2, & t_4t_5 - t_1 &= \mu_1\mu_2, \\ t_3 - t_2 &= \dot{\mu}_1 + \dot{\mu}_2, & t_2t_4 - t_3t_5 &= \dot{\mu}_1\mu_2 + \mu_1\dot{\mu}_2, \end{aligned}$$

the map is given by

$$\begin{aligned} u &= x^2 - t_6x + t_4t_5 - t_1, \\ v &= (t_2 - t_3)x + (t_2t_4 - t_3t_5). \\ w &= x^4 + t_6x^3 - (t_1 + 2t_2 + 2t_3 + t_5^2 + t_5t_6 + t_6^2)x^2 + (2(t_2t_5 + t_3t_4 - t_1t_6 + t_4t_5t_6) - t_6^3)x \\ &\quad + 4t_2t_3 + t_6(t_4t_5t_6 - t_1t_6 - 2t_2t_4 - 2t_3t_5). \end{aligned}$$

where $w(x)$ is found from the relation $u(x)w(x) + v(x)^2 = (x^3 - a_1x + b_1)^2 - 4$ (the right hand side is the Toda curve, as it appears in the linearisation). Therefore we can look at the three body Toda lattice as a subsystem of the even master system. As a by-product we get a Lax pair for the Toda lattice in *sl*(2).

Secondly recall that the seven-dimensional system linearises by the transformation

$$\begin{aligned} s_1 &= \mu_1\mu_2, & s_2 &= -\frac{\mu_1+\mu_2}{2}, \\ s_7 &= -\frac{1}{8}(\dot{\mu}_1\mu_2 + \mu_1\dot{\mu}_2), & s_5 &= -\frac{1}{8}(\dot{\mu}_1 + \dot{\mu}_2). \end{aligned}$$

It follows as for the Toda lattice that

$$\begin{aligned} u &= x^2 + 2s_2x + s_1, \\ v &= 8s_5x - 8s_7, \\ w &= x^4 - 2s_2x^3 + (s_1 - 16s_4 - 4s_2^2)x^2 - 4(s_2(s_1 - 8s_4 - 2s_2^2) + 4s_6)x \\ &\quad + 4(s_1s_2^2 + 8s_2s_6 - 16s_3), \end{aligned}$$

where w is calculated from the relation $u(x)w(x) + v(x)^2 = (x^3 + a_2x - 2b_2)^2 - 64(c_2x^2 + 2d_2x - e_2)$. This leads again to a Lax pair in $sl(2)$.

The quartic potential can also easily be related to the even master system; the transformation exhibits the invariant surfaces of the quartic potential as covers of the Abelian tori in the even master system. From the transformation

$$\begin{aligned} q_1^2 &= -\mu_1\mu_2, & q_2 &= \mu_1 + \mu_2, \\ 16p_1q_1 &= \dot\mu_1\mu_2 + \mu_1\dot\mu_2, & -32p_2 &= \dot\mu_1 + \dot\mu_2, \end{aligned}$$

it follows that

$$\begin{aligned} u &= x^2 - q_2x - q_1^2 \\ v &= 32p_2x + 16q_1p_1. \\ w &= x^4 + q_2x^3 + (q_1^2 + q_2^2)x^2 + q_2(2q_1^2 + q_2^2)x + 256p_1^2. \end{aligned}$$

As before, w is calulated from $u(x)w(x) + v(x)^2 = x^6 - 64a_3x^2 + 256b_3x$, leading to a Lax pair for the quartic potential.

Finally, combining the transformations, it follows that

$$\begin{aligned} x^2 - t_6x + t_4t_5 - t_1 &= x^2 + 2s_2x + s_1 = x^2 - q_2x - q_1^2, \\ (t_2 - t_3)x + t_2t_4 - t_3t_5 &= 8s_5x - 8s_7 = 32p_2x + 16q_1p_1, \end{aligned}$$

(the two identities coming from the polynomial w are immaterial here). From these identities we find a transformation from the three body Toda lattice as well as from the quartic potential to the seven-dimensional system (to find the expressions for s_3, s_4 and s_6 the invariants (16), (26) and (22) are used) (see [BvM]):

$$\begin{aligned} s_1 &= t_4t_5 - t_1, & s_1 &= -q_1^2, \\ s_2 &= -\frac{t_6}{2}, & s_2 &= -\frac{q_2}{2}, \\ s_3 &= -\frac{t_2t_3}{16}, & s_3 &= \frac{(q_1^2q_2^2 - 256p_1^2)}{64}, \\ s_4 &= \frac{t_2 + t_3}{8}, & s_4 &= -\frac{q_1^2 + q_2^2}{8}, \\ s_5 &= \frac{t_2 - t_3}{8}, & s_5 &= 4p_2, \\ s_6 &= \frac{t_3t_5 + t_2t_4}{8}, & s_6 &= -\frac{1}{8}q_1^2q_2, \\ s_7 &= \frac{t_3t_5 - t_2t_4}{8}, & s_7 &= -2p_1q_1. \end{aligned}$$

The Toda lattice and the quartic potential can in no way be related. The reason for this is that the curves (and hence the Jacobians) corresponding to both systems are unrelated.

References

[A] Arnold, V., *Mathematical Methods of Classical Mechanics*, Springer-Verlag (1978).

[AM] Abraham, R., Marsden, J., *Foundations of Mechanics*, Benjamin/Cummings Publishing Company (1978).

[AvM1] Adler, M., van Moerbeke, P., *Algebraic completely integrable systems: a systematic approach*, Perspectives in Mathematics, Academic Press (to appear).

[AvM2] Adler, M., van Moerbeke, P., *Representing the Kowalevski and Henon-Heiles motions as Manakov geodesic flows on $SO(4)$ and a two-dimensional family of Lax pairs*, Communications in Mathematical Physics, 114, (1987), 1-41.

[Ba] Barth, W., *Abelian surfaces with $(1,2)$ polarisation*, Conference on Algebraic Geometry, Sendai (1985).

[BRS] Bobenko, A., Reimann, A., Semenov-Tian-Shanski, *The Kowalevski top 99 Years Later: A Lax Pair, Generalisations and Explicit Solutions*, Communications in Mathematical Physics, 122, (1989), 321-354.

[BvM] Bechlivanidis, C., van Moerbeke, P., *The Goryachev-Chaplygin Top and the Toda lattice*, Communications in Mathematical Physics, 110, (1987), 317-324.

[D] Duistermaat, H., *On global action-angle coordinates*, Comm. on Pure and Appl. Math., 32, (1980), 687-706.

[F] Francoise, J., *Integrales de periode et geometrie symplectique et isochore*, Lecture Notes in Mathematics, 1416. (1990), 105-138

[FM] Flaschka, H., McLaughin, D., *Canonically conjugated variables for the KdV equation and the Toda lattice with the periodic boundary condition*, Progress in Theoretical Physics, 55, (1976), 438-456.

[GH] Griffiths, P., Harris, J., *Principles of Algebraic Geometry*, Wiley-Interscience, New-York (1978).

[H] Haine, L., *Geodesic Flow on $SO(4)$and Abelian Surfaces*, Mathematische Annalen, 263, (1983), 435-472.

[HH] Haine, L., Horozov, E., *A Lax pair for Kowalevski's top*, Physica D, 29, (1987), 173-180.

[Hu] Hudson, R.W.H., *Kummer's quartic surface*, Cambridge University Press (1990), first published in 1905.

[HvM] Horozov, E., van Moerbeke, P., *The full geometry of Kowalevski's top and (1,2)-Abelian surfaces*, Communications in Pure and Applied Mathematics, 42, 1989.

[Ko] Kowalevski, S., *Sur le problème de la rotation d'un corps solide autour d'un point fixe*, Acta Mathematica, 14, (1889), 177-232.

[LB] Lange, H., Birkenhake, Ch., *Complex Abelian Varieties*, Springer-Verlag (1992).

[MM] Magri, F., Marosi, C., *A geometrical characterization of integrable hamiltonian systems through the theory of Poisson Nijenhuis manifolds*, Publications of the Department of Mathematics of the University of Milan (1984).

[M] Mumford, D., *Tata lectures on Theta 2*, Birkhäuser (1984).

[Pi1] Piovan, L., *Algebraically completely integrable systems and Kummer varieties*, Mathematische Annalen, 290, (1991), 349-403.

[Pi2] Piovan, L., *Cyclic coverings of Abelian surfaces*, preprint.

[RDG] Ramani, A., Dorizzi, B., Grammaticos, D., *Painlevé Conjecture Revisited*, Physical review letters, 49, (1982), 1539-1541.

[VN] Veselov, A., Novikov, S., *Poisson brackets and complex tori*, Proceedings of the Steklov Institue of Mathematics, 3, (1985), 53-65.

Linearising two-dimensional integrable systems and the construction of action-angle variables

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Abstract

In this paper we show how an important class of two-dimensional integrable systems (the so-called *algebraic completely integrable systems*) can be explicitely linearised in a systematic way and how the calculation of action-angle variables derives from it. The methods will be shown to be very effective by applying them to some classical and some more recent examples. The methods will also lead in a natural way to birational maps between some of these examples, giving as a by-product a Lax pair for these systems.

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