Flag Varieties and Interpretations of Young Tableau Algorithms

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ABSTRACT

The conjugacy classes of nilpotent $n \times n$ matrices can be parametrised by partitions λ of n, and for a nilpotent η in the class parametrised by λ, the variety $\mathcal{F}_\eta$ of η-stable flags has its irreducible components parametrised by the standard Young tableaux of shape λ. We indicate how several algorithmic constructions defined for Young tableaux have significance in this context, thus extending Steinberg's result that the relative position of flags generically chosen in the irreducible components of $\mathcal{F}_\eta$ parametrised by tableaux P and Q, is the permutation associated to (P,Q) under the Robinson-Schensted correspondence. Other constructions for which we give interpretations are Schützenberger's involution of the set of Young tableaux, jeu de taquin (leading also to an interpretation of Littlewood-Richardson coefficients), and the transpose Robinson-Schensted correspondence (defined using column insertion). In each case we use a doubly indexed family of partitions, defined in terms of the flag (or pair of flags) determined by a point chosen in the variety under consideration, and we show that for generic choices, the family satisfies combinatorial relations that make it correspond to an instance of the algorithmic operation being interpreted, as described in [vLee3].

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§0. Introduction.

The Schensted algorithm, which defines a bijective correspondence between permutations and pairs of (standard) Young tableaux, and the Schützenberger (or evacuation) algorithm, which defines a shape preserving involution of the set of Young tableaux, can both be described using doubly indexed families of partitions that satisfy certain local rules, as described in [vLee3]. In this paper we show how both correspondences occur in relation to questions concerning varieties of flags stabilised by a fixed nilpotent transformation η. The mentioned doubly indexed families of partitions arise very naturally in this context, and they provide detailed information concerning the internal steps of the algorithms, rather than just about the correspondences defined by them. As a consequence, the study of the Schützenberger algorithm also leads to an interpretation of *jeu de taquin*, and of the Littlewood-Richardson coefficients. The connections between geometry and combinatorics presented here include and extend results due to Steinberg ([Stb1], [Stb2]) and Hesselink ([Hes]).

The basic fact underlying these interpretations is that the irreducible components of the variety $\mathcal{F}_\eta$ of η-stable complete flags is parametrised in a natural way by the set of standard Young tableaux of shape equal to the Jordan type $J(\eta)$ of η. In fact there are two dual parametrisations, and we show that the transition between them is given by the Schützenberger involution. Taking a projection on varieties of incomplete flags by forgetting the parts of flags below a certain dimension, we obtain from this an interpretation of jeu de taquin (operating on skew standard tableaux), and a bijection between Littlewood-Richardson tableaux and irreducible components of a variety of η-stable subspaces of fixed type and cotype.

The other interpretations involve the Robinson-Schensted algorithm and relative positions of flags. We give a derivation of Steinberg's result that the relative position of two generically chosen flags in given irreducible components of $\mathcal{F}_\eta$ is the permutation related by the Robinson-Schensted correspondence to the pair of the tableaux parametrising those components. For this result a specific choice of parametrisation is required, but there are variations of the statement for other choices; in particular when dual parametrisations are used for the two components, one obtains the transpose Robinson-Schensted correspondence, defined using column insertion. Together, these interpretations give a geometric meaning to many key properties of the combinatorial correspondences: involutivity of the Schützenberger correspondence, confluence of jeu de taquin, the fact that the number of skew tableaux of a fixed shape and given rectification P depends only on the shape of P, symmetry of the Robinson-Schensted correspondence, and the various relations between this correspondence, its transpose, and the Schützenberger correspondence.

This paper is organised as follows. In §1 we review the essential facts in the linear algebra of a vector space equipped with a nilpotent transformation η, followed in §2 by the definition of the variety $\mathcal{F}_\eta$, and the parametrisations of its irreducible components by Young tableaux. In §3 and §4 we give the respective interpretations of the Schützenberger algorithm, and of jeu de taquin and Littlewood-Richardson tableaux. In §5 we discuss relative positions of flags, which are used in §6 to give a geometric interpretation of the Robinson-Schensted correspondence, and in §7 to give a similar but independent interpretation of the transposed Robinson-Schensted correspondence.

In our combinatorial notation, as well as in our approach of the algorithms considered, we shall closely follow [vLee3]; we collect here the essential definitions used. A *partition* λ is an infinite weakly decreasing sequence $\lambda_0 \geq \lambda_1 \geq \cdots$ of natural numbers (called the *parts* of λ) with finite sum, denoted by $|\lambda|$. To each partition λ is associated its *Young diagram* $Y(\lambda) \subset \mathbf{N} \times \mathbf{N}$, defined by $(i,j) \in Y(\lambda) \iff j < \lambda_i$ (so that $\#Y(\lambda) = |\lambda|$); the *transposed* partition λ^{t} of λ is the one whose Young diagram is obtained by reflection of $Y(\lambda)$ in the main diagonal. The elements of $Y(\lambda)$ are called its *squares*, and are depicted correspondingly (so that they may be filled with values); Young diagrams are displayed with the first index i increasing downwards, the second increasing toward the right, like matrices. The set of all partitions is denoted by $\mathcal{P}$, and is partially ordered by inclusion of Young diagrams, written '$\subseteq$'; the elements of $\mathcal{P}_n = \{\, \lambda \in \mathcal{P} \mid |\lambda| = n \,\}$, are called partitions of n. For the predecessor relation in $\mathcal{P}$, which was denoted by $\mu \in \lambda^-$ in [vLee3], we shall write instead $\mu \prec \lambda$, with $\mu \preceq \lambda$ meaning that $\mu \prec \lambda$ or $\mu = \lambda$. When $\mu \prec \lambda$ and $Y(\lambda) \setminus Y(\mu) = \{x\}$ we call the square x a corner of λ and a cocorner of μ, and write $\lambda = \mu + x$, $\mu = \lambda - x$ and $\lambda - \mu = x$. We write $x \parallel y$ to indicate that squares x and y are adjacent.

A *Young tableau* is an injective map $T\colon Y(\lambda) \to \mathbf{N}$, for some $\lambda \in \mathcal{P}$ called the *shape* $\operatorname{sh} T$ of T, such that when each number $T(i,j)$ is written as entry into square (i,j), all rows and columns are increasing. Each such T determines a saturated decreasing chain $\operatorname{ch} T$ from λ to (0) in $\mathcal{P}$, by recording the successive shapes as the squares are removed from $Y(\lambda)$ in order of decreasing entries. When $\operatorname{ch} T = \operatorname{ch} T'$ we write $T \sim T'$, which is an equivalence relation on tableaux; for tableaux of fixed shape λ, a set of representatives of the equivalence classes is formed by the set $\mathcal{T}_\lambda$ of *normalised* tableaux, whose entries are all $< |\lambda|$. For any non-empty Young tableau T, the square containing the highest entry of T is denoted by $\lceil T \rceil$, and the tableau obtained by removing that square (and its entry) from T by T^-. By applying the deflation procedure used in the definition of the Schützenberger algorithm to T, a tableau $T^\downarrow$ is obtained, in which the smallest entry has disappeared. The result of applying the full Schützenberger algorithm to T is denoted by $S(T)$, and the pair of tableaux obtained by applying the Schensted algorithm to a permutation σ is denoted by $RS(\sigma)$ (see [vLee3] for definitions).

When numbering or indexing with natural numbers, these definitions all start using 0, rather than 1; this holds in particular for the parts of a partition, and the rows and columns of Young diagrams. This leads to simpler expressions, but note that while row i of $Y(\lambda)$ has length λ_i, it has no square in column λ_i. We even have gone a bit further than in [vLee3], by defining entries of normalised tableaux to start at 0. Also, the group $\mathbf{S}_n$ is taken to consist of permutations of $\{0, \ldots, n-1\}$, represented by sequences of the form $\sigma_0, \sigma_1, \ldots, \sigma_{n-1}$; in particular, the order reversing permutation $\tilde{n} \in \mathbf{S}_n$ satisfies $\tilde{n}_i = n - 1 - i$.

The work presented here was motivated by that of the author's thesis [vLee1], which deals with the significantly more complicated case of other classical groups (in characteristic $\neq 2$) instead of $\mathbf{GL}_n$. There a combinatorial algorithm analogous to the Robinson-Schensted algorithm is deduced, that performs the corresponding computation of generic relative positions of flags; however there are complications that cause the descriptions and proofs to be much more technical than those in the current paper. Our aim here was to separate the main techniques and arguments of [vLee1] from a number of distracting technicalities, by applying them in the simpler situation of $\mathbf{GL}_n$; even so the reasoning is sometimes detailed and subtle. At the same time we believe that in describing that situation as transparently as possible, we have been able to provide some new insight.

§1. Nilpotent transformations.

Let V be a vector space of finite dimension n over an infinite field k. In this section we recall some basic facts concerning V, equipped with a fixed nilpotent endomorphism η. Most of these facts are also discussed, from a somewhat more general and elevated perspective, in [Macd, Chapter II].

A subspace V' of V is called η-stable if $\eta(V') \subseteq V'$. There exists a decomposition of V as a direct sum of non-zero η-stable subspaces that cannot be so decomposed further; any summand of dimension d admits a basis $x_0, \ldots, x_{d-1}$ such that $\eta(x_0) = 0$ and $\eta(x_i) = x_{i-1}$ for $0 < i < d$. This is called a decomposition into *Jordan blocks*; it is generally not unique, but the multiset of the dimensions of the blocks depends only on η. Arranged into weakly decreasing order these dimensions form a partition of n, called the *Jordan type* $J(\eta)$ of η. Throughout this paper we write λ for $J(\eta)$ and u for the unipotent transformation $\eta + \mathbf{1} \in \mathbf{GL}(V)$ corresponding to η; when η is variable, λ and u are assumed to vary correspondingly.

One can characterise λ in terms of the powers η^j of η (among which we include $\eta^0 = \mathbf{1}$), without referring to any particular decomposition into Jordan blocks, as follows.

1.1. Proposition. *For all $c \geq 0$ one has $\dim(\ker \eta^c) = \sum_{0 \leq j < c} \lambda^{\mathrm{t}}_j$, which is the number of squares in the first c columns of the Young diagram $Y(\lambda)$. Similarly, $\dim(\operatorname{im} \eta^c) = \sum_{j \geq c} \lambda^{\mathrm{t}}_j$, which is the number of squares in the remaining columns of $Y(\lambda)$.*

Proof. This is easily verified for individual Jordan blocks, from which the general case follows. □

For any η-stable subspace V' of V, η induces nilpotent endomorphisms of the spaces V' and V/V', which will be denoted respectively by $\eta|_{V'}$ and $\eta_{/V'}$. Since a nilpotent endomorphism of a 1-dimensional space is necessarily 0, a subspace l of dimension 1 (a line) is η-stable if and only if $l \subseteq \ker \eta$, and similarly a subspace H of codimension 1 (a hyperplane) is η-stable if and only if $H \supseteq \operatorname{im} \eta$.

1.2. Proposition. *Let V' be an η-stable subspace of V. Then $J(\eta|_{V'}) \subseteq \lambda$, and $J(\eta_{/V'}) \subseteq \lambda$.*

Proof. The image of the subspace $\ker \eta^j$ under the projection $V \to V/V'$ is $\ker \eta^j / \ker(\eta|_{V'})^j$, for all $j \geq 0$; since $\ker \eta^j \subseteq \ker \eta^{j+1}$, it follows that $\dim(\ker \eta^j) - \dim(\ker(\eta|_{V'})^j)$ increases weakly as j increases:

$$\dim(\ker \eta^j) - \dim(\ker(\eta|_{V'})^j) \leq \dim(\ker \eta^{j+1}) - \dim(\ker(\eta|_{V'})^{j+1}). \tag{1}$$

The length λ^{t}_j of column j of $Y(\lambda)$ is equal to $\dim(\ker \eta^{j+1}) - \dim(\ker \eta^j)$ by proposition 1.1, so from (1) one gets $J(\eta|_{V'})^{\mathrm{t}}_j \leq \lambda^{\mathrm{t}}_j$, and combining this for all j yields $J(\eta|_{V'}) \subseteq \lambda$. Similarly, the kernel of the projection $\operatorname{im} \eta^j \to (\operatorname{im} \eta^j)/V' = \operatorname{im}(\eta_{/V'})^j$ is equal to $V' \cap \operatorname{im} \eta^j$, so its dimension $\dim(\operatorname{im} \eta^j) - \dim(\operatorname{im}(\eta_{/V'})^j)$ decreases weakly as j increases, since $\operatorname{im} \eta^j \supseteq \operatorname{im} \eta^{j+1}$. Then using $\lambda^{\mathrm{t}}_j = \dim(\operatorname{im} \eta^j) - \dim(\operatorname{im} \eta^{j+1})$, it follows analogously to the argument above that $\lambda^{\mathrm{t}}_j \geq J(\eta_{/V'})^{\mathrm{t}}_j$ for all j, whence $J(\eta_{/V'}) \subseteq \lambda$. □

The partition $J(\eta|_{V'})$ is called the type of V', and $J(\eta_{/V'})$ is the cotype of V' (in V). Since the spaces $\ker(\eta|_{V'})^j = V' \cap \ker \eta^j$ whose dimensions determine the type of V' are not directly related to the spaces $V' \cap \operatorname{im} \eta^j$ that were used to determine its cotype, it is not generally possible (for a fixed value of λ) to determine the type from the cotype or vice versa. However, there is an exception when λ is a "rectangular" partition, i.e., when all its non-zero parts have a fixed size d: in that case one has $\ker \eta^j = \operatorname{im} \eta^{d-j}$ for $0 \leq j \leq d$. This leads to the following fact.

1.3. Proposition. *If $\lambda = (d, d, \ldots, d)$ with d occurring m times, and $J(\eta|_{V'}) = (\mu_0, \ldots, \mu_{m-1})$, then $J(\eta_{/V'}) = (d - \mu_{m-1}, \ldots, d - \mu_0)$.*

Proof. From the proof of proposition 1.2, one has for $0 \leq j < d$:

$$\begin{aligned} J(\eta|_{V'})^{\mathrm{t}}_j &= \dim(V' \cap \ker \eta^{j+1}) - \dim(V' \cap \ker \eta^j) \\ &= \dim(V' \cap \operatorname{im} \eta^{d-j-1}) - \dim(V' \cap \operatorname{im} \eta^{d-j}) = \lambda^{\mathrm{t}}_{d-j-1} - J(\eta_{/V'})^{\mathrm{t}}_{d-j-1} = m - J(\eta_{/V'})^{\mathrm{t}}_{d-j-1}, \end{aligned}$$

from which the stated relation between $J(\eta|_{V'})$ and $J(\eta_{/V'})$ follows. □

We now specialise to the cases of η-stable lines and hyperplanes. As we have seen above, η-stability of a line l means $l \subseteq \ker \eta$, so the set of η-stable lines is identified with the projective space $\mathbf{P}(\ker \eta)$, which we shall denote by $\mathbf{P}(V)_\eta$. Also, the cotype $J(\eta_{/l})$ of l is determined by the (weakly decreasing) sequence of values $\dim(l \cap \operatorname{im} \eta^j) \in \{0, 1\}$, for $j \geq 0$. We define for $j \in \mathbf{N}$ subspaces

$$W_j(\eta) = \ker \eta \cap \operatorname{im} \eta^j \tag{2}$$

of $\ker \eta$, which form a weakly decreasing chain. We also define subsets

$$U_j(\eta) = \mathbf{P}(W_j(\eta)) \setminus \mathbf{P}(W_{j+1}(\eta)) \tag{3}$$

of the projective space $\mathbf{P}(V)_\eta$; the non-empty $U_j(\eta)$ form a finite partition of that space. We have $\dim W_j(\eta) = \lambda^{\mathrm{t}}_j$ by proposition 1.1; therefore $U_j(\eta)$ is non-empty if and only if the following equivalent statements hold: $\lambda^{\mathrm{t}}_j > \lambda^{\mathrm{t}}_{j+1}$; there is a corner of λ in column j; at least one part of λ equals $j+1$. For the case of η-stable hyperplanes we can apply these definitions to η^*, the nilpotent endomorphism of the dual vector space V^* given by $\eta^*(\phi)\colon v \mapsto \phi(\eta(v))$ for $\phi \in V^*$ and $v \in V$. We therefore define

$$W^*_j(\eta) = W_j(\eta^*) = \{\, \phi \in V^* \mid \phi(\operatorname{im} \eta) = 0 \wedge \phi(\ker \eta^j) = 0 \,\}. \tag{4}$$

The set $U_j(\eta^*)$ is contained in the set $\mathbf{P}(V^*)$ of 1-dimensional subspaces of V^*, which is in canonical bijection with the set of hyperplanes H of V by $H \mapsto \{\, \phi \in V^* \mid \phi(H) = 0 \,\}$. We shall denote this set of hyperplanes by $\mathbf{P}^*(V)$, and its subset $\{\, H \in \mathbf{P}^*(V) \mid H \supseteq \operatorname{im} \eta \,\}$ of η-stable hyperplanes by $\mathbf{P}^*(V)_\eta$. Then we define $U^*_j(\eta)$ as the subset of $\mathbf{P}^*(V)_\eta$ corresponding to $U_j(\eta^*)$:

$$U^*_j(\eta) = \{\, H \in \mathbf{P}^*(V)_\eta \mid H \supseteq \ker \eta^j \wedge H \not\supseteq \ker \eta^{j+1} \,\}. \tag{5}$$

The $U_j(\eta)$ and $U^*_j(\eta)$ respectively partition $\mathbf{P}(V)_\eta$ according to cotype and $\mathbf{P}^*(V)_\eta$ according to type:

1.4. Proposition. *If $l \in U_j(\eta)$, then the Young diagram of $J(\eta_{/l})$ is obtained from that of λ by removing its corner in column j. If $H \in U^*_j(\eta)$, then the Young diagram of $J(\eta|_H)$ is obtained from that of λ by removing its corner in column j.*

Proof. If $l \in U_j(\eta)$, then by reasoning as in the proof of proposition 1.2 we find that $\lambda^{\mathrm{t}}_j - J(\eta_{/V'})^{\mathrm{t}}_j = 1$, and $\lambda^{\mathrm{t}}_c = J(\eta_{/V'})^{\mathrm{t}}_c$ for all $c \neq j$. The argument for $H \in U^*_j(\eta)$ is entirely analogous. □

For any basis $\{b_0, \ldots, b_k\}$ of $\ker \eta$ with the property that each $W_j(\eta)$ is spanned by $\{\, b_i \mid 0 \leq i < \lambda^{\mathrm{t}}_j \,\}$, one can find a decomposition into Jordan blocks $V = B_0 \oplus \cdots \oplus B_k$ such that $\ker(\eta|_{B_i}) = \langle b_i \rangle$ for all i (it suffices to choose vectors v'_i with $\eta^j(v'_i) = b_i$, where j is such that$\langle b_i \rangle \in U_j(\eta)$, and to set $B_i = \langle \eta^k(v'_i) \mid 0 \leq k \leq j \rangle$). For any given $l \in \mathbf{P}(V)_\eta$ the basis can be chosen such that $l = \langle b_i \rangle$ for some i; we shall say that a corresponding decomposition of V into Jordan blocks is adapted to l. We shall similarly call a decomposition of V into Jordan blocks adapted to $H \in \mathbf{P}^*(V)_\eta$ if H contains all blocks but one; for that block B_i one has $H \cap B_i = \operatorname{im}(\eta|_{B_i})$. We see that the centraliser Z_u of u in $\mathbf{GL}(V)$ acts transitively on each set $U_j(\eta)$ and on each $U^*_j(\eta)$. The following characterisations of the index j such that $l \in U_j(\eta)$ respectively $H \in U^*_j(\eta)$ will be useful in the sequel.

1.5. Lemma.
(1) If $l \in U_j(\eta)$, then j is the minimal value for which $(\ker \eta)/l \supseteq W_j(\eta_{/l})$.
(2) If $H \in U^*_j(\eta)$, then j is the minimal value for which $\operatorname{im} \eta \subseteq \operatorname{im}(\eta|_H) + \ker(\eta|_H)^j$.
(3) $H \in U^*_j(\eta)$ if and only if $\operatorname{im} \eta + \ker \eta^j = \operatorname{im}(\eta|_H) + \ker(\eta|_H)^j$.
(4) If $H \in U^*_j(\eta)$, then $W_c(\eta|_H) = W_c(\eta)$ for $c \neq j$, while $W_j(\eta|_H)$ is a hyperplane in $W_j(\eta)$.

Proof. In each case let the initial condition be satisfied, and let $V = B_0 \oplus \cdots \oplus B_k$ be a decomposition into Jordan blocks adapted to l respectively to H. For (1), let B_i be the block containing l, then

$V/l \cong B_0 \oplus \cdots \oplus (B_i/l) \oplus \cdots \oplus B_k$, and the projection $V \to V/l$ is the identity on all summands except B_i. This reduces us to the case $V = B_i$; then $(\ker \eta)/l = \{0\}$ and $J(\eta_{/l}) = (j)$, whence the statement is obvious. For (2), let B_i be the block not contained in H, then the intersection of $\operatorname{im}\eta$ with any other block is contained in $\operatorname{im}(\eta|_H)$; this again reduces us to the case $V = B_i$, where the statement follows from $J(\eta|_H) = (j)$. Part (3) now follows because $\ker \eta^c \nsubseteq H$ for $c > j$. Part (4) also follows by considering the decomposition of V, or by observing that $W_c(\eta|_H) \subseteq W_c(\eta)$ and $\dim(W_c(\eta)) = \lambda^{\mathrm{t}}_c$, in conjunction with proposition 1.4. □

§2. Flags.

A (complete) *flag* f in V is a saturated chain $0 = f_0 \subset f_1 \subset \cdots \subset f_n = V$ of subspaces of V. We have $\dim f_i = i$, and the individual spaces f_i are called the *parts* of f. Let $\mathcal{F}$ be the set of all such flags, called the *flag manifold* of V. It has the structure of a projective algebraic variety (see [Hum, 8.1]), and the maps $f \mapsto f_i$ are morphisms onto the respective Grassmann varieties. Of particular interest are the morphisms giving the line and hyperplane parts: if $n > 0$ we define $\alpha: \mathcal{F} \to \mathbf{P}(V)$ and $\omega: \mathcal{F} \to \mathbf{P}^*(V)$ by $\alpha(f) = f_1$, $\omega(f) = f_{n-1}$. The group $\mathbf{GL}(V)$ acts on $\mathcal{F}$, $\mathbf{P}(V)$, and $\mathbf{P}^*(V)$, and clearly α and ω are $\mathbf{GL}(V)$-equivariant. We say that a flag $f \in \mathcal{F}$ is η-stable if all its parts are, and let $\mathcal{F}_\eta$ denote the subvariety of η-stable flags in $\mathcal{F}$, or equivalently the fixed point set of u acting on $\mathcal{F}$; α_η and ω_η will denote the restrictions to $\mathcal{F}_\eta$ of α and ω, respectively. We have $\operatorname{im}\alpha_\eta = \mathbf{P}(V)_\eta = \bigcup_{j\geq 0} U_j(\eta)$, and similarly $\operatorname{im}\omega_\eta = \mathbf{P}^*(V)_\eta = \bigcup_{j\geq 0} U^*_j(\eta)$.

For each η-stable hyperplane $H \in \mathbf{P}^*(V)_\eta$, the fibre $\omega_\eta^{-1}(H)$ of ω_η is isomorphic to the variety $\mathcal{F}_{\eta|_H}$ of $\eta|_H$-stable flags in H. Indeed the isomorphism is given by $f \mapsto f^-$, where $f^- = (f_0 \subset \cdots \subset f_{n-1} = H)$ is the flag obtained from f by omitting the last part $f_n = V$. Similarly, for each η-stable line $l \in \mathbf{P}(V)_\eta$, the fibre $\alpha_\eta^{-1}(l)$ is isomorphic to the variety $\mathcal{F}_{\eta_{/l}}$ of $\eta_{/l}$-stable flags in V/l; here the isomorphism will be written as $f \mapsto f^\downarrow$, where $f^\downarrow = (f_1/l \subset \cdots \subset f_n/l)$ is the flag obtained from f by reducing modulo $f_1 = l$ all its parts except f_0. There will be no confusion if the same notations f^- and $f^\downarrow$ are used when f is a flag in a vector space other than V (provided its dimension is non-zero); this allows us in particular to write for $f \in \mathcal{F}_\eta$ expressions such as f^{--}, $f^{\downarrow\downarrow\downarrow}$, and $f^{-\downarrow}$, as long as the total number of operations applied does not exceed n. Moreover, the operations commute: $f^{-\downarrow}$ and $f^{\downarrow-}$ denote the same flag in f_{n-1}/f_1. In general, one obtains in this manner from $f \in \mathcal{F}_\eta$ an η_{f_i/f_j}-stable flag in f_i/f_j, for some $i \geq j$, where η_{f_i/f_j} is the nilpotent endomorphism of f_i/f_j induced by η.

The sequences of types and of cotypes of the parts of $f \in \mathcal{F}_\eta$ define two saturated decreasing chains in $\mathcal{P}$ from λ to (0), that can be used to define Young tableaux $r_\eta(f), q_\eta(f) \in \mathcal{T}_\lambda$:

$$\operatorname{ch} r_\eta(f) = \bigl(J(\eta), J(\eta|_{f_{n-1}}), J(\eta|_{f_{n-2}}), \ldots, (0)\bigr), \tag{6}$$
$$\operatorname{ch} q_\eta(f) = \bigl(J(\eta), J(\eta_{/f_1}), J(\eta_{/f_2}), \ldots, (0)\bigr). \tag{7}$$

In other words, the subtableau of $r_\eta(f)$ containing entries $< i$ has shape $J(u|_{f_i})$, while the subtableau of $q_\eta(f)$ containing entries $< n - i$ has shape $J(u_{/f_i})$. If we define for each flag $f \in \mathcal{F}$ a dual flag f^* in V^* by ${f^*}_i = \{\, \phi \in V^* \mid \phi(f_{n-i}) = 0 \,\}$, then one readily verifies that $q_\eta(f) = r_{\eta^*}(f^*)$, and $r_\eta(f) = q_{\eta^*}(f^*)$.

As was mentioned earlier, there is in general no direct relationship between the type and cotype of the parts of f, and so there is no one-to-one correspondence between $r_\eta(f)$ and $q_\eta(f)$ either. However, we have again an exception if λ is a rectangular partition. To describe the relationship in this case, we introduce the involutive operation $T \mapsto T^\diamond$ on $\mathcal{T}_\lambda$ for rectangular λ: let the square t be the unique corner of λ, then whenever some square $s \leq t$ has entry i in t, then the diametrically opposite square $t - s$ has entry $\tilde{n}_i = n - 1 - i$ in $T^\diamond$ (this is essentially the same as the operation $P \mapsto \overline{P}$ of [vLee3, proposition 5.7]).

2.1. Proposition. *If λ is a rectangular partition, then $q_\eta(f) = r_\eta(f)^\diamond$ for all $f \in \mathcal{F}_\eta$.*

Proof. Put $\operatorname{ch} r_\eta(f) = (\lambda^n, \lambda^{n-1}, \ldots, \lambda^0)$, and $\operatorname{ch} q_\eta(f) = (\mu^n, \ldots, \mu^0)$. Then each λ^i determines μ^{n-i}, as described in proposition 1.3. For $0 \leq i < n$, the square with entry i in $r_\eta(f)$ is $\lambda^{i+1} - \lambda^i$, which determines the square $\mu^{n-i} - \mu^{n-i-1}$ containing $\tilde{n}_i$ in $q_\eta(f)$; therefore $q_\eta(f) = r_\eta(f)^\diamond$. □

We define for any Young tableau T of shape λ:

$$\mathcal{F}_{\eta,T} = \{\, f \in \mathcal{F}_\eta \mid r_\eta(f) \sim T \,\}, \tag{8}$$
$$\mathcal{F}^*_{\eta,T} = \{\, f \in \mathcal{F}_\eta \mid q_\eta(f) \sim T \,\}. \tag{9}$$

If the square $\lceil T \rceil$ lies in column j (which implies that $U_j(\eta)$ and $U^*_j(\eta)$ are non-empty), then one has $\omega(\mathcal{F}_{\eta,T}) \subseteq U^*_j(\eta)$ and $\alpha(\mathcal{F}^*_{\eta,T}) \subseteq U_j(\eta)$. Moreover, for any $H \in U^*_j(\eta)$ the fibre $\mathcal{F}_{\eta,T} \cap \omega_\eta^{-1}(H)$ is isomorphic to $\mathcal{F}_{\eta|_H,T^-}$ by $f \mapsto f^-$, and for any $l \in U_j(\eta)$ the fibre $\mathcal{F}^*_{\eta,T} \cap \alpha_\eta^{-1}(l)$ is isomorphic to $\mathcal{F}^*_{\eta/l,T^-}$ by $f \mapsto f^\downarrow$. It follows by induction that each of the sets $\mathcal{F}_{\eta,T}$ and $\mathcal{F}^*_{\eta,T}$ is non-empty, and open in its closure. As T ranges over $\mathcal{T}_\lambda$, the sets $\mathcal{F}_{\eta,T}$ partition $\mathcal{F}_\eta$ into finitely many subsets, as do the sets $\mathcal{F}^*_{\eta,T}$.

2.2. Proposition.
(1) For each $T \in \mathcal{T}_\lambda$ the sets $\mathcal{F}_{\eta,T}$ and $\mathcal{F}^*_{\eta,T}$ are irreducible.
(2) $\dim(\mathcal{F}_{\eta,T}) = \dim(\mathcal{F}^*_{\eta,T}) = n(\lambda) \stackrel{\text{def}}{=} \sum_{i\in\mathbf{N}} i\lambda_i$, regardless of $T \in \mathcal{T}_\lambda$.
(3) The set of irreducible components of $\mathcal{F}_\eta$ is equal to $\{\, \overline{\mathcal{F}_{\eta,T}} \mid T \in \mathcal{T}_\lambda \,\}$, and also to $\{\, \overline{\mathcal{F}^*_{\eta,T}} \mid T \in \mathcal{T}_\lambda \,\}$.

Proof. We proceed by induction on $n = |\lambda|$, and only prove the statements for $\mathcal{F}_{\eta,T}$, as those for $\mathcal{F}^*_{\eta,T}$ are entirely similar (and also follow by transition to the dual vector space). The case $n = 0$ is trivial, so assume $n > 0$; let $T \in \mathcal{T}_\lambda$ and let $\lceil T \rceil$ be the square (i, j). Since $U^*_j(\eta)$ is irreducible and an orbit for Z_u, it is already an orbit for the identity component Z°_u of Z_u (in fact Z_u is always connected, but we do not wish to invoke that fact here). Using the isomorphism $\mathcal{F}_{\eta,T} \cap \omega_\eta^{-1}(H) \xrightarrow{\sim} \mathcal{F}_{\eta|_H,T^-}$ for some $H \in U^*_j(\eta)$, we may define a surjective morphism $Z^\circ_u \times \mathcal{F}_{\eta|_H,T^-} \to \mathcal{F}_{\eta,T}$ by $(z, f^-) \mapsto z \cdot f$; since the domain of this morphism is irreducible by the induction hypothesis, so is its image, which establishes (1). We have $\dim(U^*_j(\eta)) = \dim(W^*_j(\eta)) - 1 = i$, and so for any $H \in U^*_j(\eta)$ we have $\dim \mathcal{F}_{\eta,T} = \dim U^*_j(\eta) + \dim \mathcal{F}_{\eta|_H,T^-} = i + n(\operatorname{sh} T^-) = n(\lambda)$, proving (2). Part (3) follows from (1) and (2). □

Remark. Part (3) gives two different natural parametrisations of the irreducible components of $\mathcal{F}_\eta$ by Young tableaux. The first one, based on the types of the parts of flags (as it uses r_η) corresponds to the parametrisation used in [Stb2], but in [Spa2, II.5.3] the other parametrisation, based on cotypes (q_η), is effectively used. We choose to work primarily with the former parametrisation, partly because it is somewhat simpler to use restrictions than quotients, but mainly because this choice leads to a more direct interpretation of the Robinson-Schensted algorithm. Note that our choice does lead to a slightly illogical use of asterisks: $\mathcal{F}_{\eta,T}$ has a fibration over $U^*_j(\eta)$, while $\mathcal{F}^*_{\eta,T}$ has one over $U_j(\eta)$.

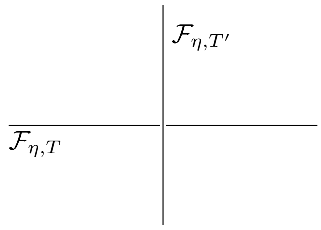
We close this section with an example, illustrating these parametrisations of the irreducible components of $\mathcal{F}_\eta$ in the simplest non-trivial case, namely for the Jordan type $\lambda = (2, 1)$. Then $\mathcal{T}_\lambda$ has 2 elements, namely

$$T = \begin{array}{|c|c|}\hline 0 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad \text{and} \qquad T' = \begin{array}{|c|c|}\hline 0 & 2 \\ \hline 1 \\ \cline{1-1}\end{array},$$

and hence $\mathcal{F}_\eta$ has 2 irreducible components, of dimension $n(\lambda) = 1$. To be specific, let us take

$$\eta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Calling the standard basis vectors e_0, e_1, e_2 we have $\ker \eta = \langle e_0, e_2 \rangle$ and $\operatorname{im} \eta = \langle e_0 \rangle$, while $\eta^j = 0$ for $j > 1$. There are two Z_u-orbits of η-stable hyperplanes, namely the set $U^*_0(\eta)$ of all hyperplanes containing $\operatorname{im} \eta = \langle e_0 \rangle$ but not $\ker \eta = \langle e_0, e_2 \rangle$, and the singleton $U^*_1(\eta) = \{\langle e_0, e_2 \rangle\}$. For any $H \in U^*_0(\eta)$ the fibre $\omega_\eta^{-1}(H)$ consists of a single flag f, with $f_1 = H \cap \ker \eta = \langle e_0 \rangle$ and of course $f_2 = H$, so $\mathcal{F}_{\eta,T}$ is isomorphic to the affine line $U^*_0(\eta)$. On the other hand for $H = \langle e_0, e_2 \rangle \in U^*_1(\eta)$ we have $\eta|_H = 0$, whence an η-stable flag f with $f_2 = H$ can have an arbitrary element of $\mathbf{P}(H)$ as f_1, so in this case the fibre $\omega_\eta^{-1}(H)$, which coincides with $\mathcal{F}_{\eta,T'}$, is a projective line. There is one element of $\mathcal{F}_{\eta,T'}$ that lies in the closure of $\mathcal{F}_{\eta,T}$, namely the flag f with $f_1 = \langle e_0 \rangle$ and $f_2 = \langle e_0, e_2 \rangle$. Therefore, the whole variety $\mathcal{F}_\eta$ can be depicted as follows



This illustration should be considered to lie in $\mathbf{P}_1 \times \mathbf{P}_1$, with the horizontal coordinate representing the choice of the hyperplane $H = f_2$, and the vertical coordinate representing the choice of the line f_1. By reasoning similar to that above, it can be seen that the fibre $\mathcal{F}^*_{\eta,T'}$ of the morphism α_η is the horizontal line *including* the point of crossing, and $\mathcal{F}^*_{\eta,T}$ is the remainder of the vertical line. So $r_\eta(f) \neq q_\eta(f)$ for all flags $f \in \mathcal{F}_\eta$, except the flag represented by the crossing point of the lines, for which $r_\eta(f) = q_\eta(f) = T' = \begin{smallmatrix}0&2\\1&\end{smallmatrix}$.

§3. Interpretation of the Schützenberger algorithm.

We shall now proceed to show that the two given parametrisations of the irreducible components of $\mathcal{F}_\eta$ are related by the involution S of $\mathcal{T}_\lambda$ defined by the Schützenberger (evacuation) algorithm. In fact we shall give a more detailed description of the situation than this. We first recall the following two kinds of configurations of 4 partitions that can occur within the doubly indexed family of partitions that was used in [vLee3, 2.2] to give a description of the Schützenberger algorithm.

3.1. Definition. *An arrangement of 4 partitions* $\left(\begin{smallmatrix}\lambda & \mu\\ \mu' & \nu\end{smallmatrix}\right)$ *with* $\nu \prec \mu \prec \lambda$ *and* $\nu \prec \mu' \prec \lambda$ *is called*

(i) a configuration of type $S1$ *if* $\mu = \mu'$ *and* $(\mu - \nu) \parallel (\lambda - \mu)$,

(ii) a configuration of type $S2$ *if* $\mu \neq \mu'$ *(so that one has* $\lambda = \mu \cup \mu'$ *and* $\nu = \mu \cap \mu'$*).*

Note that the condition $(\mu - \nu) \parallel (\lambda - \mu)$ implies $\mu = \mu'$ (there are no other partitions between ν and λ), and that in the other case the two partitions μ, μ' are the only ones between ν and λ. It follows that if $\left(\begin{smallmatrix}\lambda & \mu\\ \mu' & \nu\end{smallmatrix}\right)$ is known to be of type $S1$ or $S2$, then either one of μ or μ' is uniquely determined by the other three partitions. The relevance of these configurations becomes clear when one considers for $\nu \prec \mu \prec \lambda$ the following variety $\mathcal{E}_{\eta,\mu,\nu}$ of partial flags, consisting of only a line and hyperplane part:

$$\mathcal{E}_{\eta,\mu,\nu} = \{\, (l,H) \in \mathbf{P}(V)_\eta \times \mathbf{P}^*(V)_\eta \mid J(\eta|_H) = \mu \wedge l \subseteq H \wedge J(\eta_{H/l}) = \nu \,\}. \tag{10}$$

3.2. Lemma. *Let* $\nu \prec \mu \prec \lambda$*, and let* μ' *be determined by the condition that* $\left(\begin{smallmatrix}\lambda & \mu\\ \mu' & \nu\end{smallmatrix}\right)$ *is of type* $S1$ *or* $S2$*. Then* $\mathcal{E}_{\eta,\mu,\nu}$ *is irreducible, and* $J(\eta_{/l}) = \mu'$ *for* (l,H) *in a* Z_u*-stable dense open subset of* $\mathcal{E}_{\eta,\mu,\nu}$*.*

Proof. Put $\lambda - \mu = (i,j)$, $\mu - \nu = (r,c)$; then $\mathcal{E}_{\eta,\mu,\nu}$ consists of those pairs (l,H) with $H \in U^*_j(\eta)$ and $l \in U_c(\eta|_H)$, from which its irreducibility follows. If $j = c$ or $i = r$, then $\left(\begin{smallmatrix}\lambda & \mu\\ \mu' & \nu\end{smallmatrix}\right)$ is of type $S1$, and $J(\eta_{/l}) = \mu'$ is forced by $\nu \prec J(\eta_{/l}) \prec \lambda$. If $j \neq c$ and $i \neq r$, then $\left(\begin{smallmatrix}\lambda & \mu\\ \mu' & \nu\end{smallmatrix}\right)$ is of type $S2$; if moreover $c \neq j-1$, then $U_c(\eta|_H) = U_c(\eta)$ follows from lemma 1.5 (4), so $J(\eta_{/l}) = \lambda - (r,c) = \mu'$. In the final case that $c = j-1$ and $r > i$, we find for all $H \in U^*_j(\eta)$, again using lemma 1.5 (4), that $U_c(\eta) = \mathbf{P}(W_c(\eta)) \backslash \mathbf{P}(W_j(\eta)) \cong \mathbf{P}_r \backslash \mathbf{P}_i$ is dense and open in $U_c(\eta|_H) = \mathbf{P}(W_c(\eta)) \backslash \mathbf{P}(W_j(\eta|_H)) \cong \mathbf{P}_r \backslash \mathbf{P}_{i-1}$ (with $\mathbf{P}_{-1} = \emptyset$)), so the subset of $\mathcal{E}_{\eta,\mu,\nu}$ of pairs (l,H) for which $l \in U_c(\eta)$ is dense and open, and on this set $J(\eta_{/l}) = \lambda - (r,c) = \mu'$ holds. □

3.3. Theorem. *There is a dense open subset* $\mathcal{F}'_\eta$ *of* $\mathcal{F}_\eta$*, such that for all* $f \in \mathcal{F}'_\eta$ *the following holds: if for* $i + j \leq n$ *one puts* $\lambda^{[i,j]} = J(\eta_{f_{n-j}/f_i})$*, then any configuration*

$$\begin{pmatrix} \lambda^{[i,j]} & \lambda^{[i,j+1]} \\ \lambda^{[i+1,j]} & \lambda^{[i+1,j+1]} \end{pmatrix}$$

is either of type $S1$ *or of type* $S2$*.*

Proof. It will suffice to find for each $T \in \mathcal{T}_\lambda$ a dense open subset of $\mathcal{F}_{\eta,T}$ on which the stated condition holds. We use induction on n, with $n = 1$ as trivial starting case. Fix $T \in \mathcal{T}_\lambda$, and let the family $(\lambda^{[i,j]})_{i+j \leq n}$ of partitions be defined by $\operatorname{ch} T = (\lambda^{[0,0]}, \lambda^{[0,1]}, \ldots, \lambda^{[0,n]})$, and by the condition on configurations in the statement of the theorem; we shall show that for appropriately chosen $f \in \mathcal{F}_{\eta,T}$ one has $J(\eta_{f_{n-j}/f_i}) = \lambda^{[i,j]}$ for $i + j \leq n$. Choose some $H \in \omega(\mathcal{F}_{\eta,T})$, then by the induction hypothesis applied to $\mathcal{F}_{\eta|_H,T^-}$, we have for f in a dense open subset of the fibre $\mathcal{F}_{\eta,T} \cap \omega_\eta^{-1}(H)$ that all instances of $J(\eta_{f_{n-j}/f_i}) = \lambda^{[i,j]}$ with $j > 0$ hold. Since $\omega(\mathcal{F}_{\eta,T})$ is equal to some Z_u-orbit $U_c^*(\eta)$, this property extends by the action of Z_u to a Z_u-stable dense open subset $\mathcal{U}$ of $\mathcal{F}_{\eta,T}$. By construction $\mathcal{U}$ is contained in $\{ f \in \mathcal{F}_{\eta,T} \mid (f_1, f_{n-1}) \in \mathcal{E}_{\eta,\mu,\nu} \}$, which projects onto $\mathcal{E}_{\eta,\mu,\nu}$ with fibres that are all of dimension $n(\lambda^{[1,1]})$. It follows that $\mathcal{U}$ meets the inverse image under this projection of the dense open subset of $\mathcal{E}_{\eta,\mu,\nu}$ described in lemma 3.2. The intersection $\mathcal{U}'$ of $\mathcal{U}$ with this inverse image is a Z_u-stable dense open subset of $\mathcal{F}_{\eta,T}$ on which the additional property $J(\eta_{/f_1}) = \lambda^{[1,0]}$ holds. Then $\alpha(\mathcal{U}')$ is a single Z_u-orbit, and $\mathcal{U}'$ intersects each fibre $\alpha_\eta^{-1}(l)$ for $l \in \alpha(\mathcal{U}')$ densely; the image of this intersection under $f \mapsto f^\downarrow$ is a dense open subset of $\mathcal{F}_{\eta_{/l},T^\downarrow}$. The proof can now be completed by applying the induction hypothesis to $\mathcal{F}_{\eta_{/l},T^\downarrow}$. □

3.4. Corollary. *For all $T \in \mathcal{T}_\lambda$ one has $\overline{\mathcal{F}_{\eta,T}} = \overline{\mathcal{F}^*_{\eta,S(T)}}$.*

Proof. This follows immediately from the theorem, and the fact that the family $\lambda^{[i,j]}$ with the mentioned properties can be used to compute $S(T)$ (cf. the proof of [vLee3, theorem 2.2.1]). □

The proof given for this geometric interpretation uses a description of S from which it obvious that S is an involution (in fact one that was used to prove this fact combinatorially). However, even if we forget the proof, the interpretation clearly implies that S is an involution, since $f \in \mathcal{F}^*_{\eta,T}$ is equivalent to $f^* \in \mathcal{F}_{\eta^*,T}$, and $f^{**} = f$. In combination with proposition 2.1, the corollary also implies that $S(T) \sim T^\diamond$ for tableaux T of rectangular shape, as stated in [vLee3, Corollary 5.7]; in this case the geometric proof has no purely combinatorial counterpart (the proof given in [vLee3] is based on the relationship of the Schützenberger correspondence with the Robinson-Schensted correspondence). There is on the other hand one combinatorially obvious symmetry of S without any clear geometric meaning, namely the fact that it commutes with transposition (a geometric interpretation of this fact would require some operation that gives rise to transposition of Jordan types). Nevertheless, this symmetry of S will be important below, when we give geometric interpretations of the Robinson-Schensted correspondence and its transpose.

§4. Interpretation of jeu de taquin, and Littlewood-Richardson tableaux.

The deflation procedure used in the Schützenberger algorithm is related to the operation of jeu de taquin (also known as *glissement*), which is performed on skew tableaux. Jeu de taquin can be described completely in terms of the deflation procedure [vLee3, §5], which allows us to deduce from theorem 3.3 an interpretation of jeu de taquin. That theorem does not mention tableaux or their entries directly however, but is stated in terms of families of partitions, and for our interpretation, all that matters about a skew tableau is the chain in $\mathcal{P}$ associated to it. For convenience we shall work directly with such chains, rather than with skew tableaux: we define a skew chain of shape λ/μ to be a saturated decreasing chain in $\mathcal{P}$ from λ to μ. We denote the set of all skew chains of shape λ/μ by $\mathcal{T}_{\lambda/\mu}$; the set $\mathcal{T}_\lambda$ is in bijection with $\mathcal{T}_{\lambda/\emptyset}$ by $P \mapsto \operatorname{ch} P$.

We proceed to give a geometric interpretation to skew chains, in analogy to the definition of $\mathcal{F}_{\eta,T}$ for Young tableaux T. Let $\mathcal{G}^\eta_m$ be the variety of m-dimensional η-stable subspaces of V, and let $\mathcal{F}^{(m)}_\eta$ be the set of η-stable partial flags in V with parts in dimensions m and higher, i.e., of chains $f = (f_m \subset f_{m+1} \subset \cdots \subset f_n)$ with $f_i \in \mathcal{G}^\eta_i$. For $f \in \mathcal{F}^{(m)}_\eta$ the part f_m of minimal dimension will be denoted by $\lfloor f \rfloor$, and we define the complete flag $\overline{f} = (f_m/\lfloor f \rfloor \subset \cdots \subset f_n/\lfloor f \rfloor) \in \mathcal{F}_{\eta/\lfloor f \rfloor}$ by reducing all parts of f modulo $\lfloor f \rfloor$. We also define $r_\eta(f) = \bigl(J(\eta), J(\eta|_{f_{n-1}}), \ldots, J(\eta|_{\lfloor f \rfloor})\bigr) \in \mathcal{T}_{\lambda/\mu}$ where $\mu = J(\eta|_{\lfloor f \rfloor})$, and put

$$\mathcal{F}_{\eta,K} = \{ f \in \mathcal{F}^{(m)}_\eta \mid r_\eta(f) = K \}. \tag{11}$$

4.1. Proposition. *$\mathcal{F}_{\eta,K}$ is an irreducible variety for every $K \in \mathcal{T}_{\lambda/\mu}$, and $\dim(\mathcal{F}_{\eta,K}) = n(\lambda) - n(\mu)$.*

Proof. The proof is entirely analogous to that of parts (a) and (b) of proposition 2.2, the only difference being that the induction starts at $|\lambda| = |\mu|$ rather than at $|\lambda| = 0$. □

For a Young tableau $T \in \mathcal{T}_\lambda$, let $T_{<m}$ denote its subtableau of entries less than m; putting $\mu = \operatorname{sh} T_{<m}$, let $T_{\geq m} \in \mathcal{T}_{\lambda/\mu}$ denote the subchain $(\operatorname{sh} T, \operatorname{sh} T^-, \ldots, \mu)$ of $\operatorname{ch} T$ corresponding to the remaining entries. If we denote by $T^{\downarrow * i}$ the result of applying the deflation procedure i times to T, then the relation $K \rhd K'$ of glissement (K' is obtainable from K by inward jeu de taquin slides) is defined in [vLee3, §5] by restricting T and $T^{\downarrow * i}$ to skew subtableaux with the same set of entries. The corresponding definition for skew chains is that $T_{\geq m} \rhd T^{\downarrow * i}{}_{\geq m}$ for $i \leq m$. We obtain from theorem 3.3:

4.2. Corollary. *Let $K \in \mathcal{T}_{\lambda/\mu}$ be a skew chain, and let $P \in \mathcal{T}_\nu$ be a Young tableau such that $K \triangleright \operatorname{ch} P$. Then there is a dense open subset $\mathcal{F}'_{\eta,K}$ of $\mathcal{F}_{\eta,K}$ such that $r_{\eta/\lfloor f \rfloor}(\overline{f}) = P$ for all $f \in \mathcal{F}'_{\eta,K}$.*

Proof. Since $K \triangleright \operatorname{ch} P$, there exists a $T \in \mathcal{T}_\lambda$ with $K = T_{\geq m}$ and $P \sim T^{\downarrow * m}$. Let p denote the natural projection $\mathcal{F}_{\eta,T} \to \mathcal{F}_{\eta,K}$ (which is clearly surjective), and take $\mathcal{F}'_{\eta,K} = p(\mathcal{F}_{\eta,T} \cap \mathcal{F}'_\eta)$, where $\mathcal{F}'_\eta$ is as in theorem 3.3; the conclusion of that theorem immediately implies $r_{\eta/\lfloor f \rfloor}(\overline{f}) = P$ for $f \in \mathcal{F}'_{\eta,K}$. □

In this geometric interpretation, confluence of jeu de taquin is obvious: P is uniquely determined by K. We now partition $\mathcal{G}^\eta_m$ according to the type and cotype of its elements: for $\mu \in \mathcal{P}_m$, $\nu \in \mathcal{P}_{n-m}$ put

$$\mathcal{G}^\eta_{\mu,\nu} = \{\, X \in \mathcal{G}^\eta_m \mid J(\eta|_X) = \mu \wedge J(\eta_{/X}) = \nu \,\}; \tag{12}$$

furthermore, denote by $\mathcal{T}^{\triangleright\nu}_{\lambda/\mu}$ the set of $K \in \mathcal{T}_{\lambda/\mu}$ such that $K \triangleright \operatorname{ch} P$ for some $P \in \mathcal{T}_\nu$.

4.3. Theorem. *Denote by $\pi: \mathcal{F}^{(m)}_\eta \to \mathcal{G}^\eta_m$ the morphism defined by $\pi(f) = \lfloor f \rfloor$. Let $\mu \in \mathcal{P}_m$ and $\nu \in \mathcal{P}_{n-m}$.*

(1) If $K \in \mathcal{T}^{\triangleright\nu}_{\lambda/\mu}$, then the image $\pi(\mathcal{F}'_{\eta,K})$ of the set $\mathcal{F}'_{\eta,K}$ of corollary 4.2 is dense in an irreducible component of $\mathcal{G}^\eta_{\mu,\nu}$ of dimension $n(\lambda;\mu,\nu) \stackrel{\text{def}}{=} n(\lambda) - n(\mu) - n(\nu)$. Moreover, any such component is so obtained as $\pi(\overline{\mathcal{F}_{\eta,K}}) \cap \mathcal{G}^\eta_{\mu,\nu}$ for at least one $K \in \mathcal{T}^{\triangleright\nu}_{\lambda/\mu}$, and there are no components of higher dimension.

(2) The number of irreducible components of $\mathcal{G}^\eta_{\mu,\nu}$ of dimension $n(\lambda;\mu,\nu)$ is equal to the Littlewood-Richardson coefficient $c^\lambda_{\mu,\nu}$, and they can be explicitly parametrised by the Littlewood-Richardson tableaux of shape λ/μ and weight ν.

(3) For $K, L \in \mathcal{T}^{\triangleright\nu}_{\lambda/\mu}$, the irreducible components $\pi(\overline{\mathcal{F}_{\eta,K}}) \cap \mathcal{G}^\eta_{\mu,\nu}$ and $\pi(\overline{\mathcal{F}_{\eta,L}}) \cap \mathcal{G}^\eta_{\mu,\nu}$ of $\mathcal{G}^\eta_{\mu,\nu}$ are equal if and only if K and L are dual equivalent in the sense of [Haim].

Limiting itself to what can be deduced from corollary 4.2, the theorem avoids any statement about possible irreducible components of $\mathcal{G}^\eta_{\mu,\nu}$ of dimension less than $n(\lambda;\mu,\nu)$. However, we shall show below that such components do not exist, and so an accordingly simplified and strengthened form of the theorem does in fact hold.

Proof. (1) It is clear that $\pi^{-1}(\mathcal{G}^\eta_{\mu,\nu}) \subseteq \bigcup_{K \in \mathcal{T}_{\lambda/\mu}} \mathcal{F}_{\eta,K}$, whose components have dimension $n(\lambda) - n(\mu)$, while the fibre $\pi^{-1}(X)$ at any $X \in \mathcal{G}^\eta_{\mu,\nu}$ is isomorphic to $\mathcal{F}_{\eta/X}$, whose components have dimension $n(\nu)$. Therefore any irreducible component C of $\mathcal{G}^\eta_{\mu,\nu}$ can have dimension at most $n(\lambda;\mu,\nu)$, and when it has this dimension, $\pi^{-1}(C)$ is dense in some union of sets $\mathcal{F}_{\eta,K}$ with $K \in \mathcal{T}_{\lambda/\mu}$. By corollary 4.2 one has $\pi(\mathcal{F}'_{\eta,K}) \subseteq \mathcal{G}^\eta_{\mu,\nu}$ whenever $K \in \mathcal{T}^{\triangleright\nu}_{\lambda/\mu}$, from which the claims follow.

(2) Corollary 4.2 also implies that if $K \triangleright \operatorname{ch} P$, then $\mathcal{F}_{\eta,K}$ meets any fibre $\pi^{-1}(X)$ with $X \in \pi(\mathcal{F}'_{\eta,K})$ in the irreducible component of that fibre that corresponds to $\mathcal{F}_{\eta/X,P}$; hence, fixing an arbitrary $P \in \mathcal{T}_\nu$, the irreducible components of $\mathcal{G}^\eta_{\mu,\nu}$ of dimension $n(\lambda;\mu,\nu)$ correspond bijectively to the skew chains $K \in \mathcal{T}^{\triangleright\nu}_{\lambda/\mu}$ with $K \triangleright \operatorname{ch} P$. The number of such K is known to be independent of the choice of P ([Schü2, (3.7)]) and equal to $c^\lambda_{\mu,\nu}$ ([Schü2, (4.7)], see also [vLee4, Theorem 5.2.5]). In fact there is a specific $P \in \mathcal{T}_\nu$ for which the Littlewood-Richardson tableaux T of shape λ/μ and weight ν correspond directly to the skew chains $K \in \mathcal{T}^{\triangleright\nu}_{\lambda/\mu}$ with $K \triangleright \operatorname{ch} P$. To associate a skew chain K to a semistandard skew tableau T, one uses the well known process of standardisation: its sequence of partitions, starting at λ, is found by removing the squares of T by decreasing entries, and among squares with equal entries from right to left. Jeu de taquin is defined for semistandard tableaux in such a way that it commutes with this standardisation, and it preserves the property of being a Littlewood-Richardson tableau. The indicated special tableau $P \in \mathcal{T}_\nu$ is such that $\operatorname{ch} P$ is the standardisation of the tableau $\mathbf{T}_\nu$ of shape ν in which each row i is filled with entries i ($\mathbf{T}_\nu$ is the unique Littlewood-Richardson tableau of shape ν, and it has weight ν). Then $K \in \mathcal{T}_{\lambda/\mu}$ is the standardisation of a Littlewood-Richardson tableau T of weight ν if and only if $K \triangleright \operatorname{ch} P$.

(3) Define for $K, L \in \mathcal{T}^{\triangleright\nu}_{\lambda/\mu}$ the equivalence relation $K \equiv L$ to mean $\pi(\overline{\mathcal{F}_{\eta,K}}) \cap \mathcal{G}^\eta_{\mu,\nu} = \pi(\overline{\mathcal{F}_{\eta,L}}) \cap \mathcal{G}^\eta_{\mu,\nu}$. We have established above that for any $P \in \mathcal{T}_\nu$ the jeu de taquin equivalence class $\{\, K \in \mathcal{T}^{\triangleright\nu}_{\lambda/\mu} \mid K \triangleright \operatorname{ch} P \,\}$ is a set of representatives of the classes for '$\equiv$'. As the members of such a jeu de taquin equivalence class are mutually dual inequivalent (this is the easy part of [Haim, theorem 2.13]), it will suffice to prove

that $K \equiv L$ implies the dual equivalence of K and L. We shall establish this by finding a sequence of jeu de taquin slides that transforms K and L respectively into $K', L' \in \mathcal{T}_{\nu/\emptyset}$, preserving equality of shapes at each step; being (chains corresponding to) Young tableaux of the same shape, K' and L' will be dual equivalent [Haim, corollary 2.5], which implies dual equivalence of K and L. Since $K \equiv L$, it is possible to choose partial flags $f \in \mathcal{F}'_{\eta,K}$ and $f' \in \mathcal{F}'_{\eta,L}$ with $\lfloor f \rfloor = \lfloor f' \rfloor$; by extending f and f' identically with suitably chosen parts in dimensions less than m, one obtains flags $\hat{f}, \hat{f}' \in \mathcal{F}'_\eta$ such that $T = r_\eta(\hat{f})$ and $U = r_\eta(\hat{f}')$ satisfy $T_{\geq m} = K$, $U_{\geq m} = L$, and $T_{<m} = U_{<m}$. Then for $i \leq m$ the shapes of the skew chains $T^{\downarrow * i}{}_{\geq m}$ and $U^{\downarrow * i}{}_{\geq m}$ are both equal to $J(\eta_{/\hat{f}_i})/J(\eta_{\lfloor f \rfloor/\hat{f}_i})$ (since $\hat{f}_i = \hat{f}'_i$), which gives the required sequence of slides transforming K and L respectively into $K' = \operatorname{ch} T^{\downarrow * m}$ and $L' = \operatorname{ch} U^{\downarrow * m}$. □

The detailed statement of the theorem appears to be new. However, the relation between $\mathcal{G}^\eta_{\mu,\nu}$ and Littlewood-Richardson coefficients was already indicated in [Spr1, Theorem 4.4]. The setting there is in fact more general, with a semi-simple linear algebraic group G replacing $\mathbf{GL}_n$; correspondingly, $\mathcal{G}^\eta_{\mu,\nu}$ is replaced by a variety $\mathcal{X}'_{A,B}(P)$ of parabolic subgroups, and Littlewood-Richardson coefficients by decomposition multiplicities for representations induced from the Weyl group W' of a Levi factor of P to the Weyl group W of G. (Our theorem corresponds only to maximal parabolic P, but can be extended easily so as to correspond to arbitrary parabolic subgroups.) The number of irreducible components of $\mathcal{G}^\eta_{\mu,\nu}$ of dimension $n(\lambda;\mu,\nu)$ appears in a somewhat disguised form, as a decomposition multiplicity $n_{A,B,\phi,\psi}$ for a permutation action of a group $C_G(A) \times C_G(B)$ on the set of irreducible components of $\mathcal{X}'_{A,B}(P)$ of that dimension; for $\mathbf{GL}_n$, the group $C_G(A) \times C_G(B)$ is always trivial, and $n_{A,B,\phi,\psi}$ reduces to the number of components acted (trivially) upon.

Another, rather weaker, connection between the Littlewood-Richardson rule and the geometry of $\mathcal{F}_\eta$ was indicated in [Sri]. Its main theorem (4.2) corresponds to our corollary 4.2, but it is stated (and proved) in a somewhat roundabout fashion in terms of permutations, whose link to geometry is formed by Steinberg's interpretation of the Robinson-Schensted correspondence (which will be discussed below). In fact, that theorem itself involves no geometry at all, and it can be proved in a purely combinatorial manner. Since only a fixed maximal parabolic subgroup P is considered, no connection with the geometry of $\mathcal{G}^\eta_{\mu,\nu}$ is indicated; the Littlewood-Richardson coefficients, which arise in relation to jeu de taquin in the same way as above, are only given their traditional representation theoretic interpretation.

As we have seen, dual equivalence classes in $\mathcal{T}^{\triangleright\nu}_{\lambda/\mu}$ are in bijection with Littlewood-Richardson tableaux of shape λ/μ and weight ν. Once the latter have all been determined, one can construct the set $\{\, K \in \mathcal{T}^{\triangleright\nu}_{\lambda/\mu} \mid K \triangleright \operatorname{ch} P \,\}$ effectively, not just for the special tableau P indicated in the proof above, but for any given $P \in \mathcal{T}_\nu$. Such a construction is given in the proof of [vLee4, Theorem 5.2.5] in terms of the Robinson-Schensted correspondence for "pictures"; we shall formulate it here without using pictures. One associates to any skew chain $K \in \mathcal{T}_{\lambda/\mu}$ a permutation $w(K)$ by concatenating the rows of the skew tableau corresponding to K, taking them in order from bottom to top. Call the two Young tableaux $(P,Q) = RS(w(K))$ the P-symbol and Q-symbol of K; then the P-symbol of K characterises its jeu de taquin equivalence class (indeed $K \triangleright \operatorname{ch} P$), and the Q-symbol its dual equivalence class. Whenever $c^\lambda_{\mu,\nu} > 0$, all tableaux in $\mathcal{T}_\nu$ occur as P-symbol of some $K \in \mathcal{T}^{\triangleright\nu}_{\lambda/\mu}$, but not necessarily as Q-symbol. The set of tableaux that do so occur, is precisely the set $Q(\lambda/\mu,\nu)$ of Q-symbols of Littlewood-Richardson tableaux of shape λ/μ and weight ν (where the Q-symbol of a semistandard skew tableau is defined either as the Q-symbol of its standardisation, or directly by concatenating its rows and applying the version of the Schensted algorithm that allows repeated entries; either way the Q-symbol is a standard tableau). One then has

$$\{\, w(K) \mid K \in \mathcal{T}^{\triangleright\nu}_{\lambda/\mu} \wedge K \triangleright \operatorname{ch} P \,\} = \{\, RS^{-1}(P,Q) \mid Q \in Q(\lambda/\mu,\nu) \,\}, \tag{13}$$

from which the desired set of skew chains K is readily reconstructed.

Now as promised we shall rule out the possibility that $\mathcal{G}^\eta_{\mu,\nu}$ could have irreducible components of dimension less than $n(\lambda;\mu,\nu)$, which implies in particular that $\mathcal{G}^\eta_{\mu,\nu} = \emptyset$ whenever $c^\lambda_{\mu,\nu} = 0$. This requires an algebraic construction that associates a Littlewood-Richardson tableau to any individual element $X \in \mathcal{G}^\eta_{\mu,\nu}$. We essentially use the construction described in [Macd, II 3], but since our context is dual to the one considered there, we shall present an adapted version of the construction and proof.

4.4. Proposition. *For any $\lambda,\mu,\nu\in\mathcal{P}$, the irreducible components of $\mathcal{G}^\eta_{\mu,\nu}$ are precisely those described in theorem 4.3, i.e., $\mathcal{G}^\eta_{\mu,\nu}$ has no irreducible components of dimension less than $n(\lambda;\mu,\nu)$.*

Proof. We shall construct for any $X\in\mathcal{G}^\eta_{\mu,\nu}$ a tableau $K\in\mathcal{T}^{\trianglerighteq\nu}_{\lambda/\mu}$ with $X\in\pi(\mathcal{F}_{\eta,K})$; then X lies in the component $\pi(\overline{\mathcal{F}_{\eta,K}})\cap\mathcal{G}^\eta_{\mu,\nu}$, and the proposition follows. Fix $X\in\mathcal{G}^\eta_{\mu,\nu}$, and for $i\in\mathbf{N}$ put $X_i=\eta^{-i}(X)$ and $\mu^i=J(\eta|_{X_i})$ (in particular $\mu^0=\mu$ and $\mu^{\nu_0}=\lambda$). By filling each skew diagram $Y(\mu^{i+1})\setminus Y(\mu^i)$ with entries i we obtain the transpose of a semistandard tableau T of shape $\lambda^{\mathrm{t}}/\mu^{\mathrm{t}}$ and weight ν^{t} (by proposition 1.2, since $J(\eta_{/X})=\nu$); we claim that T is a Littlewood-Richardson tableau (i.e., $T\trianglerighteq\mathbf{T}_{\nu^{\mathrm{t}}}$). Assuming this for the moment, the transposes $K\in\mathcal{T}_{\lambda/\mu}$ and $P\in\mathcal{T}_{\nu/\emptyset}$ of the standardisations of T and $\mathbf{T}_{\nu^{\mathrm{t}}}$ satisfy $K\trianglerighteq P$ (jeu de taquin commutes with transposition), so $K\in\mathcal{T}^{\trianglerighteq\nu}_{\lambda/\mu}$. To show that $X\in\pi(\mathcal{F}_{\eta,K})$, it suffices to extend the sequence of subspaces $X_0\subset X_1\subset\cdots\subset X_{\nu_0}$ by interpolation to some $f\in\mathcal{F}_{\eta,K}$. Now η acts as 0 on each of the quotient spaces X_{i+1}/X_i, so any choice of complete flags in those spaces leads to a $f\in\mathcal{F}^{(m)}_\eta$, which has moreover the property that all partitions μ^i occur in the chain $r_\eta(f)$; we only need to show that it is possible to obtain $r_\eta(f)=K$. In fact, among the irreducible set of choices for f, a dense subset has $r_\eta(f)=K$; this follows from the observation that for any subspace $V'\supseteq\operatorname{im}\eta$, the projective space $S=\{\,H\in\mathbf{P}^*(V)_\eta\mid H\supseteq V'\,\}$ meets $U^*_j(\eta)$ whenever the vertical strip $Y(\lambda)\setminus Y(J(\eta|_{V'}))$ meets column j, and then of course the intersection is dense in S for the minimal such j.

It remains to show that T is a Littlewood-Richardson tableau. That T is a semistandard tableau means that each $Y(\mu^{i+1})\setminus Y(\mu^i)$ is a vertical strip, or equivalently $(\mu^{i+1})^{\mathrm{t}}_c\geq(\mu^i)^{\mathrm{t}}_c\geq(\mu^{i+1})^{\mathrm{t}}_{c+1}$ for $i,c\in\mathbf{N}$; this follows from the easily verified inclusions $W_c(\eta|_{X_{i+1}})\supseteq W_c(\eta|_{X_i})\supseteq W_{c+1}(\eta|_{X_{i+1}})$. The remaining conditions for T to be a Littlewood-Richardson tableau can be formulated in several equivalent ways, but the following will be practical here, in view of proposition 1.1: denoting by $T(i,c)$ the number of entries i in the first c rows of T, one should have $T(i+1,c+1)\leq T(i,c)$ for $i,c\in\mathbf{N}$ (this implies $T(i,c)=0$ when $i\geq c$). Now $T(i,c)$ is the difference between the number of squares in the first c columns of $Y(\mu^{i+1})$ and of $Y(\mu^i)$; putting $X^c_i=\ker(\eta|_{X_i})^c=X_i\cap\ker\eta^c$, we therefore have by proposition 1.1 that $T(i,c)=\dim X^c_{i+1}-\dim X^c_i=\dim(X^c_{i+1}/X^c_i)$. Now for all i,c one has $\eta^{-1}(X^c_i)=X^{c+1}_{i+1}$, whence η induces an injective map $X^{c+1}_{i+2}/X^{c+1}_{i+1}\to X^c_{i+1}/X^c_i$, giving the required inequality $T(i+1,c+1)\leq T(i,c)$. □

§5. Relative positions of flags.

Besides the use mentioned above of the Robinson-Schensted algorithm as a computational aid in dealing with classes of jeu de taquin equivalence and dual equivalence, there is also a direct geometric interpretation, due to Steinberg, of the correspondence defined by it. To formulate it, we need to attach a geometric meaning to permutations; this meaning will be derived from the fact that permutations of n parametrise the orbits for the diagonal action of $\mathbf{GL}_n$ on $\mathcal{F}\times\mathcal{F}$. This parametrisation can be defined by associating to each pair (f,f') of flags a permutation called the *relative position* of f and f' that will characterise the $\mathbf{GL}_n$-orbit of (f,f'). By definition $\sigma\in\mathbf{S}_n$ is the relative position of f and f' if there exists a basis $e_0,\ldots,e_{n-1}$ of V such that $f_i=\langle e_0,\ldots,e_{i-1}\rangle$ and $f'_i=\langle e_{\sigma_0},\ldots,e_{\sigma_{i-1}}\rangle$ for all $i\leq n$. The fact that there always exists a unique such σ is the essence of Bruhat's lemma for $\mathbf{GL}_n$, but it is useful to give here an explicit demonstration of this fact. We shall use the auxiliary concept of a growth matrix.

5.1. Definition. *A growth matrix of order n is a matrix $(A_{i,j})_{0\leq i,j\leq n}$ with entries in $\mathbf{N}$ satisfying*

- *a.* $A_{i,0}=A_{0,i}=0$ *and* $A_{i,n}=A_{n,i}=i$ *for* $0\leq i\leq n$,
- *b.* $A_{i+1,j}-A_{i,j}\in\{0,1\}$ *and* $A_{i,j+1}-A_{i,j}\in\{0,1\}$ *for* $0\leq i,j<n$,
- *c.* $A_{i+1,j+1}=A_{i,j+1}\Rightarrow A_{i+1,j}=A_{i,j}$ *for* $0\leq i,j<n$ *(or equivalently,* $A_{i+1,j+1}=A_{i+1,j}\Rightarrow A_{i,j+1}=A_{i,j}$*).*

Growth matrices of order n correspond bijectively to permutations of n: for a growth matrix $A=(A_{i,j})_{0\leq i,j\leq n}$, the corresponding permutation $\sigma(A)\in\mathbf{S}_n$ has a permutation matrix Π (given by $\Pi_{i,j}=\delta_{i,\sigma(A)_j}$ for $0\leq i,j<n$) that is related to A by the equivalent relations

$$\Pi_{i,j}=A_{i+1,j+1}-A_{i+1,j}-A_{i,j+1}+A_{i,j}\qquad\text{for }0\leq i,j<n,\tag{14}$$

$$A_{i,j}=\sum_{0\leq i'<i}\ \sum_{0\leq j'<j}\Pi_{i',j'}\qquad\text{for }0\leq i,j\leq n.\tag{15}$$

We define $\pi(f,f') = \sigma(A)$, where the growth matrix A is given by $A_{i,j} = \dim(f_i \cap f'_j)$. From the definition it follows that if σ is the relative position of f and f', then $\pi(f,f') = \sigma$; in particular σ is unique. Conversely, for $\sigma = \pi(f,f')$, a basis $e_0,\ldots,e_{n-1}$ witnessing the fact that σ is the relative position of f and f' can be constructed: for each i put $j = \sigma_i^{-1}$ (so that $\Pi_{i,j} = 1$) and choose for e_i any vector in the complement of the subspace $f_i \cap f'_j$ within $f_{i+1} \cap f'_{j+1}$ (by the construction of σ this subspace is a hyperplane, and equal to both $f_{i+1} \cap f'_j$ and $f_i \cap f'_{j+1}$). As an example, if $f = f'$, then $A_{i,j} = \min(i,j)$, whence $\pi(f,f')$ is the identity permutation; at the other extreme, when f and f' are in general position one has $A_{i,j} = \max(0, i+j-n)$, whence $\pi(f,f') = \tilde{n}$, the order reversing permutation.

Remark. Besides the mentioned growth matrix A, growth matrices B, C, and D can also be associated to (f,f') with $B_{i,j} = \dim(f_i/(f_i \cap f'_{n-j}))$, $C_{i,j} = \dim(f'_j/(f_{n-i} \cap f'_j))$, and $D_{i,j} = \dim(V/(f_{n-i} + f'_{n-j}))$. Putting $\sigma = \sigma(A) = \pi(f,f')$, it can easily be verified that $\sigma(B) = \sigma\tilde{n}$, $\sigma(C) = \tilde{n}\sigma$, and $\sigma(D) = \tilde{n}\sigma\tilde{n}$. Since $\dim(V/(f_{n-i}+f'_{n-j})) = \dim(f^*_i \cap f'^*_j)$, the last case implies that the relative position of the dual flags is obtained by conjugation by $\tilde{n}$: $\pi(f^*, f'^*) = \tilde{n}\pi(f,f')\tilde{n}$. It is also obvious that $\pi(f',f) = \pi(f,f')^{-1}$.

Remark. The Bruhat order '$\leq$' on $\mathbf{S}_n$ can be defined in terms the associated growth matrices: if $\sigma = \sigma(A)$ and $\sigma' = \sigma(A')$, then one has $\sigma \leq \sigma'$ if and only if $A_{i,j} \geq A'_{i,j}$ for all i,j. It follows that for any $\sigma \in \mathbf{S}_n$, the closure of $\{\,(f,f') \mid \pi(f,f') = \sigma\,\}$ in $\mathcal{F} \times \mathcal{F}$ is $\{\,(f,f') \mid \pi(f,f') \leq \sigma\,\}$.

§6. Interpretation of the Robinson-Schensted algorithm.

In this section we shall demonstrate the result, due to Steinberg, that for a pair of flags generically chosen in irreducible components of $\mathcal{F}_\eta$ parametrised by a pair of standard Young tableaux, their relative position is related to those tableaux by the Robinson-Schensted correspondence. Like the interpretation of the Schützenberger correspondence, this will be deduced from a more detailed statement, that gives an interpretation of all the partitions in the doubly indexed family describing the algorithm; in the current case that family is the one of [vLee3, 3.2]. We recall the basic configurations that can occur.

6.1. Definition. *An arrangement of 4 partitions* $\left(\begin{smallmatrix}\nu & \mu\\ \mu' & \lambda\end{smallmatrix}\right)$ *is called*
(i) a configuration of type $RS1$ *if* $\nu = \mu = \mu' \prec \lambda$ *and* $\mu_0 = \lambda_0 - 1$,
(ii) a configuration of type $RS2$ *if* $\nu \prec \mu = \mu' \prec \lambda$, *with* $\mu_{i+1} = \lambda_{i+1} - 1$ *and* $\nu_i = \mu_i - 1$ *for some* $i \geq 0$,
(iii) a configuration of type $RS3$ *if* $\nu \prec \mu \prec \lambda$, $\nu \prec \mu' \prec \lambda$, *and* $\mu \neq \mu'$,
(iv) a configuration of type $RS0$ *if* $\nu = \mu \preceq \mu' = \lambda$ *or* $\nu = \mu' \preceq \mu = \lambda$.

6.2. Definition. *A family of partitions* $(\lambda^{[i,j]})$, *where* i *and* j *each range over an interval of* $\mathbf{Z}$, *is said to be of type* RS *if any configuration* $\left(\begin{smallmatrix}\lambda^{[i,j]} & \lambda^{[i,j+1]}\\ \lambda^{[i+1,j]} & \lambda^{[i+1,j+1]}\end{smallmatrix}\right)$ *is of one of the types* $RS0$–$RS3$.

Families of type RS were used in the proof of [vLee3, theorem 3.2.1] to relate a permutation σ and the pair $(P,Q) = RS(\sigma)$ of tableaux computed from it by the Robinson-Schensted algorithm. The pair (P,Q) describes the partitions at the boundary of the family, while σ describes the places where a configuration of type $RS1$ occurs; specifying either of these suffices to determine the entire family. More precisely, to a family $(\lambda^{[i,j]})_{0\leq i,j\leq n}$ of type RS with $\lambda^{[n,n]} = \lambda$, we associate tableaux $P,Q \in \mathcal{T}_\lambda$ such that $\operatorname{ch} P = (\lambda^{[n,n]}, \lambda^{[n-1,n]}, \ldots, \lambda^{[0,n]})$ and $\operatorname{ch} Q = (\lambda^{[n,n]}, \lambda^{[n,n-1]}, \ldots, \lambda^{[n,0]})$, and a permutation $\sigma = \sigma(A)$ where A is the growth matrix with $A_{i,j} = |\lambda^{[i,j]}|$, which means that $i = \sigma_j$ if and only if $\left(\begin{smallmatrix}\lambda^{[i,j]} & \lambda^{[i,j+1]}\\ \lambda^{[i+1,j]} & \lambda^{[i+1,j+1]}\end{smallmatrix}\right)$ is of type $RS1$. (In its relation to P, Q, and σ, our family has its indices interchanged with respect to [vLee3]; it remains of type RS, and it still establishes the relation $(P,Q) = RS(\sigma)$.)

6.3. Theorem. *Let* $P,Q \in \mathcal{T}_\lambda$, *and* $\sigma \in \mathbf{S}_n$ *with* $(P,Q) = RS(\sigma)$. *Then for all* (f,f') *in a dense open subset of* $\mathcal{F}_{\eta,P} \times \mathcal{F}_{\eta,Q}$, *the family* $J(\eta|_{f_i \cap f'_j})_{0\leq i,j\leq n}$ *is of type* RS, *and in particular* $\pi(f,f') = \sigma$.

6.4. Lemma. *Let* $\mu,\mu' \prec \lambda$, *and let* ν *be determined by the condition that* $\left(\begin{smallmatrix}\nu & \mu\\ \mu' & \lambda\end{smallmatrix}\right)$ *is of one of the types* $RS1$–$RS3$. *For any* $H_0 \in \mathbf{P}^*(V)_\eta$ *with* $J(\eta|_{H_0}) = \mu'$, *there is a dense open subset of* $\{\,H \in \mathbf{P}^*(V)_\eta \mid J(\eta|_H) = \mu\,\}$ *on which* $J(\eta|_{H\cap H_0}) = \nu$ *holds.*

Proof. If $\mu \neq \mu'$ the configuration is of type $RS3$, and it follows from proposition 1.2 that one always has $J(\eta|_{H\cap H_0}) = \nu = \mu \cap \mu'$. Assume now that $\mu = \mu'$, and let $\lambda - \mu = (i,j)$, so that H ranges over the set $U^*_j(\eta)$, which has dimension i. If $i = 0$ the configuration is of type $RS1$ and $U^*_j(\eta) = \{H_0\}$, so that for the unique choice $H = H_0$ one has $J(\eta|_{H\cap H_0}) = J(\eta|_H) = \mu = \nu$. Finally, if $i > 0$, the configuration is of type $RS2$. We claim that intersection with H_0 defines a surjective morphism $\overline{U_j^*(\eta)} \setminus \{H_0\} \to \overline{U_{j'}^*(\eta|_{H_0})}$, where $\mu - \nu = (i-1, j')$: the lemma then follows by taking for the dense open subset of $U_j^*(\eta)$ the intersection of $U_j^*(\eta)$ with the inverse image of $U_{j'}^*(\eta|_{H_0})$. To prove the claim, observe that restriction to H_0 defines a linear map $W_j^*(\eta) \to W_j^*(\eta|_{H_0}) = W_{j'}^*(\eta|_{H_0})$, the surjectivity of which follows from lemma 1.5 (3), or from a dimension consideration. □

We note that in the final case each of the fibres of the indicated map $\overline{U_j^*(\eta)} \setminus \{H_0\} \to \overline{U_{j'}^*(\eta|_{H_0})}$ meets $U_j^*(\eta)$, whence the restriction of that map to $U_j^*(\eta) \setminus \{H_0\}$ is still surjective. This means that in this case any partition obtained from μ by removing a corner in some column $c > j'$ can arise as $J(\eta|_{H \cap H_0})$ for some non-generic $H \in U_j^*(\eta)$; of course, by taking $H = H_0$ one can obtain $J(\eta|_{H \cap H_0}) = \mu$ as well.

The lemma can also be formulated as follows: putting $(i,j) = \lambda - \mu$ and $(i',j') = \mu' - \nu$, one has for H in a dense subset of $U_j^*(\eta)$ that $H \cap H_0 \in U_{j'}^*(\eta|_{H_0})$. It is not true in general, however, that as H traverses this subset of $U_j^*(\eta)$, the values $H \cap H_0$ traverse a dense subset of $U_{j'}^*(\eta|_{H_0})$. The reason for this is that within H_0 equipped with $\eta|_{H_0}$, the subspace $\operatorname{im}\eta$ is in no way special (it is not fixed under automorphisms), yet $H \cap H_0$ always contains it (because both H_0 and H do so). Although in some cases it can be shown that all hyperplanes in $U_{j'}^*(\eta|_{H_0})$ contain $\operatorname{im}\eta$, there are also cases where this is not so. This circumstance makes the proof of theorem 6.3 more difficult than that of theorem 3.3. There are however no difficulties in extending lemma 6.4 as follows.

6.5. Lemma. *Fix an arbitrary flag $f' \in \mathcal{F}_\eta$ and $P \in \mathcal{T}_\lambda$; for all f in a dense open subset of $\mathcal{F}_{\eta,P}$, the subfamily of $J(\eta|_{f_i \cap f'_j})_{0 \le i,j \le n}$ determined by $j \in \{n-1, n\}$ is of type RS.*

Clearly only the hyperplane part $H = f'_{n-1}$ of f' is relevant here. The sequence of subspaces $f_i \cap H$ forms a complete $\eta|_H$-stable flag in H, except that one part is repeated. Denoting this flag (without the repetition) by $f \cap H$, the lemma states that for generic $f \in \mathcal{F}_{\eta,P}$ one has $f \cap H \in \mathcal{F}_{\eta|_H,P'}$ and $f_{i_0} \cap H = f_{i_0+1} \cap H$, where P' and i_0 are found by applying Schensted extraction (reverse insertion) to P, starting at the square $\lambda - J(\eta|_H)$ (recall that the entries of P start at 0).

Proof. We apply induction on n. Put $(\lambda^n, \ldots, \lambda^0) = \operatorname{ch} P$ (so that $\lambda^i = J(\eta|_{f_i})$), and $\mu^i = J(\eta|_{f_i \cap H})$. Applying lemma 6.4 with $\mu' = \mu^n$, $\mu = \lambda^{n-1}$, and $H_0 = H$, we get for f_{n-1} in a dense open subset $\mathcal{U}$ of $\omega(\mathcal{F}_{\eta,P})$ that $\left(\begin{smallmatrix} \mu^{n-1} & \lambda^{n-1} \\ \mu^n & \lambda^n \end{smallmatrix}\right)$ is of one of the types $RS1$–$RS3$. If it is of type $RS1$, then $f_{n-1} = H$ for all $f \in \mathcal{F}_{\eta,P}$, so that $\mu^i = \lambda^i$ for $i < n$, and the remaining configurations $\left(\begin{smallmatrix} \mu^i & \lambda^i \\ \mu^{i+1} & \lambda^{i+1} \end{smallmatrix}\right)$ $(0 \le i < n-1)$ are of type $RS0$. Otherwise we fix any $K \in \mathcal{U}$ and consider the fibre $\omega^{-1}(K)$; the remaining configurations are then taken care of by induction applied to $\eta|_K$ and $K \cap H$ in place of η and H, respectively. □

This lemma does not suffice as an induction step for the proof of theorem 6.3. The induction hypothesis will describe the generic relative position of the pair formed by a flag in $\mathcal{F}_{\eta|_H,P'}$ and f'^-, but for reasons indicated above, the set of flags $f \cap H$ is not generally dense in $\mathcal{F}_{\eta|_H,P'}$, and it is conceivable that the generic relative position of $f \cap H$ and f'^- is different (smaller in the Bruhat order). In order to dispell this possibility, we shall show that it is possible to write generic flags $f \in \mathcal{F}_{\eta|_H,P'}$ as $\hat{f} \cap H$ for some $\hat{f} \in \mathcal{F}_{\hat\eta,P}$, where $\hat\eta$ is another nilpotent transformation of V than η, but with $J(\hat\eta) = \lambda$ and $\hat\eta|_H = \eta|_H$. To that end, we shall now consider the reverse process of restricting a nilpotent transformation to a hyperplane. Let $0 \le i_0 \le n$ and define the following vector bundle over $\mathcal{F}_{\eta,T}$:

$$\mathcal{F}_{\eta,T,i_0} = \{\, (f,v) \in \mathcal{F}_{\eta,T} \times V \mid v \in f_{i_0} \,\}. \tag{16}$$

Clearly $\mathcal{F}_{\eta,T,i_0}$ is stable under the diagonal Z_u-action on $\mathcal{F}_{\eta,T} \times V$. For any $(f,v) \in \mathcal{F}_{\eta,T,i_0}$ we define in the $n+1$-dimensional space $V \times k$ a nilpotent transformation $\hat\eta$ and a flag $\hat f$, as follows. Let $e = (0_V, 1) \in V \times k$, then $\hat\eta$ is determined by $\hat\eta|_V = \eta$ and $\hat\eta(e) = v$, and $\hat f$ is defined by

$$\hat f_i = \begin{cases} f_i & \text{if } 0 \le i \le i_0, \\ f_{i-1} \oplus \langle e \rangle & \text{if } i_0 \le i-1 \le n. \end{cases} \tag{17}$$

It follows from $v \in f_{i_0}$ that $\hat f$ is $\hat\eta$-stable.

6.6. Lemma. *Let $T \in \mathcal{T}_\lambda$ and $0 \le i_0 \le n$. For all (f,v) in a dense open subset of $\mathcal{F}_{\eta,T,i_0}$, the associated $\hat\eta$ and $\hat f$ satisfy the following property: with $\nu^i = J(\hat\eta|_{\hat f_i})$ and $\lambda^i = J(\eta|_{\hat f_i \cap V})$ for $0 \le i \le n+1$, any configuration of the form $\left(\begin{smallmatrix}\lambda^i & \nu^i \\ \lambda^{i+1} & \nu^{i+1}\end{smallmatrix}\right)$ is of type $RS0$ if $i < i_0$, of type $RS1$ if $i = i_0$, and of type $RS2$ or $RS3$ if $i > i_0$.*

Proof. One has $\operatorname{ch} T = (\lambda^{n+1}, \ldots, \lambda^{i_0+1} = \lambda^{i_0}, \ldots, \lambda^0)$, since $\hat f_i \cap V$ is f_i if $i \le i_0$, and f_{i-1} otherwise. Then the unique sequence of partitions ν^i for which the given conditions on configurations hold is readily constructed; we shall take this as definition of ν^i, and show that $J(\hat\eta|_{\hat f_i}) = \nu^i$ holds for $0 \le i \le n+1$, provided (f,v) is chosen in an appropriate set. The statement for $i < i_0$ is immediate. For the remaining statements we shall apply induction on n (fixing i_0), starting at $n = i_0$. Put $\nu = \nu^{n+1}$ and let $\nu - \lambda = (r,c)$; then by lemma 1.5 (2), the statement $J(\hat\eta) = \nu$ means that c is the minimal value with $v \in \operatorname{im}\eta + \ker\eta^c$. For $n = i_0$ one has $\mathcal{F}_{\eta,T,i_0} = \mathcal{F}_{\eta,T} \times V$, $r = 0$, and $c = \nu^{\mathrm{t}}_0$, which is the least value for which $\operatorname{im}\eta + \ker\eta^c = V$ (in fact $\ker\eta^c = V$), so the subset of $\mathcal{F}_{\eta,T,i_0}$ of pairs (f,v) for which $J(\hat\eta) = \nu$ is dense and open.

As preparation for the induction step, note that the statement of the lemma implies that the smallest subspace of V containing $\operatorname{im}\eta$ and all values v for generic (and hence for all) $(f,v) \in \mathcal{F}_{\eta,T,i_0}$, which is necessarily Z_u-stable, equals $\operatorname{im}\eta + \ker\eta^c$. Now assume $n > i_0$, and choose an arbitrary hyperplane $H \in \omega(\mathcal{F}_{\eta,T})$. Defining $\hat\omega\colon \mathcal{F}_{\eta,T,i_0} \to \mathbf{P}^*(V)$ by $\hat\omega(f,v) = \omega(f)$, it will suffice to show that the property in the lemma holds in a dense open subset of $\hat\omega^{-1}(H) \cong \mathcal{F}_{\eta|_H,T^-,i_0}$, as one can then use the action of Z_u to extend this subset to one of $\mathcal{F}_{\eta,T,i_0}$. The induction hypothesis gives a dense open subset $\mathcal{U}$ of $\hat\omega^{-1}(H)$ on which $J(\hat\eta|_{\hat f_i}) = \nu^i$ for $0 \le i \le n$; it remains to show that generically one also has $J(\hat\eta) = \nu$. If $\nu^n \ne \lambda$ then this holds on all of $\mathcal{U}$ by proposition 1.2. Otherwise $\left(\begin{smallmatrix}\lambda^n & \nu^n \\ \lambda & \nu\end{smallmatrix}\right)$ is of type $RS2$; let $\nu^n - \lambda^n = \lambda - \lambda^n = (r-1, c')$. By the induction hypothesis, the smallest subspace of H containing $\{\, v \mid (f,v) \in \mathcal{U} \,\}$ is $\operatorname{im}(\eta|_H) + \ker(\eta|_H)^{c'}$. Moreover, since $H \in U^*_{c'}(V)$ this is equal to $\operatorname{im}\eta + \ker\eta^{c'}$ by lemma 1.5 (3), and since $c = \lambda_r$ is the largest part $\le c'$ of λ, also to $\operatorname{im}\eta + \ker\eta^c$. If $c = 0$ one has $J(\hat\eta) = \nu$ on all of $\mathcal{U}$; otherwise this holds on the open subset $\{\, (f,v) \in \mathcal{U} \mid v \notin \operatorname{im}\eta + \ker\eta^{c-1} \,\}$, which is non-empty (since $\lambda_r = c$ implies $\dim(\operatorname{im}\eta + \ker\eta^{c-1}) < \dim(\operatorname{im}\eta + \ker\eta^c)$) and therefore dense. □

We have now collected all the ingredients necessary to proceed with an inductive proof of theorem 6.3.

Proof of theorem 6.3. For $n < 2$ the theorem is obvious, so assume $n \ge 2$; we shall prove that for any choice of $H \in \omega(\mathcal{F}_{\eta,Q})$, the family $J(\eta|_{f_i \cap f'_j})_{0 \le i,j \le n}$ is of type RS, for (f,f') in a dense open subset of $\varphi(H) = \mathcal{F}_{\eta,P} \times (\omega^{-1}(H) \cap \mathcal{F}_{\eta,Q})$. Let P' and i_0 be the result of applying Schensted extraction to P starting at square $\lambda^{[n,n]} - \lambda^{[n,n-1]}$, so that $\operatorname{ch}(P') = (\lambda^{[n,n-1]}, \ldots, \lambda^{[i_0+1,n-1]} = \lambda^{[i_0,n-1]}, \ldots, \lambda^{[0,n-1]})$. Lemma 6.5 provides a dense open subset $\mathcal{U}'$ of $\varphi(H)$ on which the configurations $\left(\begin{smallmatrix}\lambda^{[i-1,n-1]} & \lambda^{[i-1,n]} \\ \lambda^{[i,n-1]} & \lambda^{[i,n]}\end{smallmatrix}\right)$ are all of the required types, which implies $f \cap H \in \mathcal{F}_{\eta|_H,P'}$. On the other hand the induction hypothesis provides a dense open subset $\mathcal{U}''$ of $\mathcal{F}_{\eta|_H,P'} \times \mathcal{F}_{\eta|_H,Q^-}$ such that remaining configurations $\left(\begin{smallmatrix}\lambda^{[i-1,j-1]} & \lambda^{[i-1,j]} \\ \lambda^{[i,j-1]} & \lambda^{[i,j]}\end{smallmatrix}\right)$ are of the required types whenever $(f \cap H, f'^-) \in \mathcal{U}''$. (Here the family of partitions obtained for (P', Q^-) has been enlarged by duplicating each $\lambda^{[i_0,j]}$, to account for the fact that $f_{i_0} \cap H = f_{i_0+1} \cap H$.) The set $\mathcal{U} = \{\, (f,f') \in \mathcal{U}' \mid (f \cap H, f'^-) \in \mathcal{U}'' \,\}$ is open in $\varphi(H)$, and it remains to show that $\mathcal{U} \ne \emptyset$.

The set $\mathcal{U}''$ and the dense open set of lemma 6.6, applied with $\eta|_H$ for η and P' for T, both project to a dense open subset of $\mathcal{F}_{\eta|_H,P'}$, and the intersection of these images is non-empty. For a choice of inverse images of any point in the intersection, one finds a nilpotent transformation $\hat\eta$ of $H \times k$ with $J(\hat\eta) = \lambda$ and $\hat\eta|_H = \eta_H$, and a pair $(\hat f, f') \in \mathcal{F}_{\hat\eta,P} \times \mathcal{F}_{\hat\eta,Q}$ with $J(\hat\eta|_{\hat f_i \cap f'_j}) = \lambda^{[i,j]}$ for $0 \le i,j \le n$. Since $J(\hat\eta) = J(\eta)$ there exists a linear isomorphism $g\colon H \times k \to V$ for which $g \circ \hat\eta = \eta \circ g$, and by the transitive action of Z_u on $\omega(\mathcal{F}_{\eta,Q})$ one can achieve that moreover $g(H) = H$; then $g(\hat f, f') \in \mathcal{U}$ so that $\mathcal{U} \ne \emptyset$, which completes the proof. □

Remark. Note that in the final argument we do not assume that g fixes H pointwise; indeed, this may not be possible, since it would imply $\operatorname{im}\hat\eta = g(\operatorname{im}\hat\eta) = \operatorname{im}\eta$ (the latter identity follows from $g \circ \hat\eta = \eta \circ g$), but the whole point of the construction of $\hat\eta$ was to avoid fixing $\operatorname{im}\hat\eta$ to any particular subspace of H.

§7. Interpretation of the transposed Robinson-Schensted algorithm.

In this section we deduce, in close analogy to the previous section, an interpretation of the version of the Robinson-Schensted algorithm that is defined using column insertion instead of row insertion, and hence yields transposes of the tableaux P and Q. We start with a statement that is dual to lemma 1.5 (4).

7.1. Lemma. *Let $l \in U_c(\eta)$, and let $\iota\colon (V/l)^* \to V^*$ be the map induced by the canonical projection $V \to V/l$. For $j \neq c$ one has $\iota(W_j^*(\eta_{/l})) = W_j^*(\eta)$, while $\iota(W_c^*(\eta_{/l}))$ has codimension 1 in $W_c^*(\eta)$.*

Proof. Clearly ι is injective, and the image of $\operatorname{im}\eta + \ker\eta^j$ under the projection $V \to V/l$ is contained in $\operatorname{im}\eta_{/l} + \ker(\eta_{/l})^j$, whence $\iota(W_j^*(\eta_{/l})) \subseteq W_j^*(\eta)$. The lemma follows by dimension consideration. □

7.2. Definition. *For any arrangement of partitions $\left(\begin{smallmatrix}\nu & \mu\\ \kappa & \lambda\end{smallmatrix}\right)$ of type RSi ($i = 0, \ldots, 3$), the corresponding arrangement of transposed partitions $\left(\begin{smallmatrix}\nu^{\mathrm{t}} & \mu^{\mathrm{t}}\\ \kappa^{\mathrm{t}} & \lambda^{\mathrm{t}}\end{smallmatrix}\right)$ is called a configuration of type $RS^{\mathrm{t}}i$ (the types $RS^{\mathrm{t}}0$ and $RS^{\mathrm{t}}3$ are identical to types $RS0$ and $RS3$, respectively). If a family of partitions $(\lambda^{[i,j]})$ is of type RS, then the family of transposed partitions is said to be of type RS^{t}.*

7.3. Lemma. *Let $\mu, \mu' \prec \lambda$, and let ν be determined by the condition that $\left(\begin{smallmatrix}\nu & \mu\\ \mu' & \lambda\end{smallmatrix}\right)$ is of one of the types $RS^{\mathrm{t}}1$–$RS^{\mathrm{t}}3$. Then for any $l \in \mathbf{P}(V)_\eta$ with $J(\eta_{/l}) = \mu'$, there is a dense open subset of $\{\, H \in \mathbf{P}^*(V)_\eta \mid J(\eta|_H) = \mu \,\}$ on which $J(\eta_{H/(l\cap H)}) = \nu$ holds.*

Proof. If the configuration is of type $RS^{\mathrm{t}}3$, the conclusion follows (without the need to restrict to a dense subset) by proposition 1.2. Otherwise $\mu = \mu'$, and we put $\lambda - \mu = (r, c)$, so that H ranges over the set $U_c^*(\eta)$ (of dimension r). By lemma 7.1, the subspace $\iota(W_c^*(\eta_{/l}))$ of $W_c^*(\eta)$, has codimension 1; therefore, while $H \supseteq \ker\eta^c$ for all $H \in U_c^*(\eta)$, there is a dense open subset $\mathcal{U}$ on which $H \not\supseteq \eta^{-c}(l)$. If the configuration is of type $RS^{\mathrm{t}}1$ (i.e., $c = 0$), one has $l \cap H = \{0\}$ for $H \in \mathcal{U}$, and therefore $J(\eta_{H/(l\cap H)}) = \mu = \nu$. Finally if $c > 0$, so that the configuration is of type $RS^{\mathrm{t}}2$, it follows from $H \supseteq \ker\eta$ or $l \in \operatorname{im}\eta$ that $H \supseteq l$; for $H \in \mathcal{U}$ one has $H/l \not\supseteq \ker(\eta_{/l})^c$, but $H/l \supseteq \ker(\eta_{/l})^{c-1}$ by application of lemma 7.1 for $j = c - 1$, whence $H/l \in U_{c-1}^*(\eta_{/l})$. Then $J(\eta_{H/l})$ differs from $\mu = J(\eta_{/l})$ by a square in column $c - 1$, and is therefore equal to ν. □

Note that the generic cases match those of Lemma 6.4, with all partitions transposed, but the non-generic cases do not. We know of no geometric explanation for the first fact, but the latter is no surprise, since transposition reverses the dominance ordering on Jordan types that describes orbit closures.

7.4. Lemma. *Let $l \in \mathbf{P}(V)_\eta$. For all flags f in a dense open subset of $\mathcal{F}_\eta$, the following holds: defining $\lambda^i = J(\eta|_{f_i})$ and $\mu^i = J(\eta_{f_i/l\cap f_i})$ for $0 \le i \le n$, any configuration of the form $\left(\begin{smallmatrix}\mu^i & \lambda^i\\ \mu^{i+1} & \lambda^{i+1}\end{smallmatrix}\right)$ is of one of the types $RS^{\mathrm{t}}0$–$RS^{\mathrm{t}}3$.*

Proof. This follows from lemma 7.3 just like lemma 6.5 follows from 6.4. □

Similarly to the situation for the interpretation of the ordinary Robinson-Schensted correspondence, the sequence of spaces $f_i/(l \cap f_i)$ forms a complete $\eta_{/l}$-stable flag in V/l with one part repeated. For generic $f \in \mathcal{F}_{\eta,T}$ this flag lies in $\mathcal{F}_{\eta_{/l},T'}$ and has part i repeated, where this time T' and i can be found by applying the Schensted column extraction procedure to T starting at square $\lambda - J(\eta_{/l})$, but again the set of flags so obtained is not generally dense in the indicated set. Here too the difficulty can be resolved by a converse construction to (in this case) dividing out the line l.

For $0 \le i_0 \le n$ and define the following Z_u-stable set (a vector bundle over $\mathcal{F}_{\eta,T}$):

$$\mathcal{F}_{\eta,T,i_0^*} = \{\, (f, \phi) \in \mathcal{F}_{\eta,T} \times V^* \mid f_{i_0} \subseteq \ker\phi \,\}. \tag{18}$$

To $(f, \phi) \in \mathcal{F}_{\eta,T,i_0^*}$ we associate in the $n+1$-dimensional space $V \times k$ a nilpotent transformation $\tilde\eta$ and a flag $\tilde f$, as follows. For $(v, x) \in V \times k$ put $\tilde\eta(v, x) = (\eta(v), \phi(v))$, and with $l = \{0\} \times k \subseteq V \times k$ define

$$\tilde f_i = \begin{cases} f_i & \text{if } 0 \le i \le i_0, \\ f_{i-1} \oplus l & \text{if } i_0 \le i - 1 \le n. \end{cases} \tag{19}$$

Since $f_{i_0} \subseteq \ker\phi$, the flag $\tilde f$ is $\tilde\eta$-stable. We identify $(V \times k)/l$ with V, so that $\tilde f_i/l \cong f_{i-1}$ for $i > i_0$.

7.5. Lemma. *Let $T \in \mathcal{T}_\lambda$, and $0 \le i_0 \le n$. For all (f, ϕ) in a Z_u-stable dense open subset of $\mathcal{F}_{\eta,T,i_0^*}$, the associated $\tilde\eta$ and $\tilde f$ satisfy the following: with $\nu^i = J(\tilde\eta|_{\tilde f_i})$ and $\lambda^i = J(\eta_{\tilde f_i/(l\cap f_i)})$ for $0 \le i \le n + 1$, any configuration of the form $\left(\begin{smallmatrix}\lambda^i & \nu^i\\ \lambda^{i+1} & \nu^{i+1}\end{smallmatrix}\right)$ is of type $RS^{\mathrm{t}}0$ if $i < i_0$, of type $RS^{\mathrm{t}}1$ if $i = i_0$, and of type $RS^{\mathrm{t}}2$ or $RS3$ if $i > i_0$.*

Proof. The proof is analogous to that of lemma 6.6, with variations as noted below. Let the families ν^i and λ^i be determined by the requirements on the configurations; put $\nu = \nu^{n+1}$ and $\nu - \lambda = (r, c)$.

Since $(\ker\tilde\eta)/l \cong \ker\eta\cap\ker\phi$, the statement $J(\tilde\eta)=\nu$ means that $c=\min\{\,j \mid W_j(\eta)\subseteq\ker\phi\,\}$, by lemma 1.5 (1). In the starting case $n=i_0$ of the induction this holds, as $c=0$ and $(f,\phi)\in\mathcal{F}_{\eta,T,n^*}$ implies $\phi=0$. For $n>i_0$ and $H\in\omega(\mathcal{F}_{\eta,T})$ there is a surjection $(f,\phi)\mapsto(f^-,\phi|_H)$ of the fibre $\tilde\omega^{-1}(H)$ (with $\tilde\omega(f,\phi)=\omega(f)$) onto the variety $\mathcal{F}_{\eta|_H,T^-,i_0^*}$ (not an isomorphism, as in the proof of lemma 6.6). The induction hypothesis provides dense open subset of $\mathcal{F}_{\eta|_H,T^-,i_0^*}$; let $\mathcal{U}\subseteq\tilde\omega^{-1}(H)$ be its inverse image, and $\nu^n-\lambda^n=(r',c')$, then $W_{c'}(\eta|_H)=\ker(\eta|_H)\cap\bigcap_{(f,\phi)\in\mathcal{U}}\ker(\phi|_H)$. In the interesting case $RS^{\mathrm t}2$ one has $\nu^n=\lambda$ and $c'=c+1$, whence $W_{c'}(\eta)$ strictly contains $W_{c'}(\eta|_H)$ by lemma 1.5 (4), and $W_{c'}(\eta)\not\subseteq\ker\phi$ for some $(f,\phi)\in\mathcal{U}$. Therefore $\{\,(f,\phi)\in\mathcal{U}\mid W_{c'}(\eta)\not\subseteq\ker\phi\,\}$ is dense in $\mathcal{U}$, and on this subset $c=\min\{\,j\mid W_j(\eta)\subseteq\ker\phi\,\}$, since $W_c(\eta)=W_c(\eta|_H)\subseteq\ker\phi$ for all $(f,\phi)\in\tilde\omega^{-1}(H)$. □

7.6. Theorem. *Let $P,Q\in\mathcal{T}_\lambda$, and $\sigma\in\mathbf{S}_n$ with $(P^{\mathrm t},Q^{\mathrm t})=RS(\sigma)$. Then for all (f,f') in a dense open subset of $\mathcal{F}_{\eta,P}\times\mathcal{F}^*_{\eta,Q}$, the family $J(\eta_{f_i/(f_i\cap f'_{n-j})})_{0\le i,j\le n}$ is of type $RS^{\mathrm t}$, and in particular $\pi(f,f')\tilde n=\sigma$.*

Proof. The proof is analogous to that of theorem 6.3, using lemmas 7.4 and 7.5 instead of 6.5 and 6.6. □

Note that using theorem 3.3, the equivalence of the descriptions of the generic value of $\pi(f,f')$ given in theorems 6.3 and 7.6 follows from the purely combinatorial statement [vLee3, theorem 4.1.1]. We have preferred to give independent proofs of theorems 6.3 and 7.6, thereby giving a geometric explanation for the "witchcraft operating behind the scenes" (cf. [Kn2, p. 60]) of that combinatorial theorem.

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