An application of Hopf-Algebra techniques to representations of finite Classical Groups

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§1. Introduction.

Zelevinsky has shown in [Zelev] how the representation theory of the symmetric groups S_n, and of finite linear groups $GL_n(\mathbf{F}_q)$ can be elegantly described—with respect to the operations of induction and restriction, repectively parabolic induction and restriction—using the language of Hopf algebras, or more precisely of what he calls *Positive Selfadjoint Hopf-algebras* or *P.S.H.-algebras*. Their combinatorial properties can be deduced from a limited set of axioms. In this paper we propose a similar approach to the representation theory of other finite classical groups, namely the symplectic, unitary and orthogonal groups over $\mathbf{F}_q$. The corresponding graded groups of characters will be given a module and comodule structure over the P.S.H.-algebra $R(q)$ corresponding to $GL_n(\mathbf{F}_q)$.

In the case of $GL_n(\mathbf{F}_q)$ the Hopf-axiom which expresses the relationship between the algebra and coalgebra structure of $R(q)$ is of fundamental importance. Similarly our first goal is to determine the relationship between the two structures of module and comodule over $R(q)$. Naïvely one would hope to obtain the structure of a "Hopf-module", but it turns out that the axiom in question is not satisfied, but rather that a more complicated relation holds. In the first part of this paper (§§2,3) we introduce the basic concepts used, define the classical groups considered and the morphisms on the associated graded groups that establish the module and comodule structure, and determine their relation to one another.

The second part of this paper (§§4,5) we axiomatise the properties found, calling the objects thus characterised twisted P.S.H.-modules, and then proceed to develop a structure theory for them. This is done along the same lines followed in [Zelev], namely reducing the general case to that of 'elementary' modules, and then proving the essential uniqueness of the latter. A qualitative difference is that, using the results of Zelevisnky, we can easily construct basic examples of elementary modules, so we do not have to discover the structure of elementary modules at the same time that we prove their uniqueness. Nevertheless a substantial amount of combinatorial computation is required, of the same general nature as in [Zelev §§3,4]. The result is a parametrisation of the representations of the finite classical groups considered, and a description of the operations of 'parabolic' induction and restriction; everything in terms of the cuspidal representations, which are the atomic objects for the structure considered.

§2. Recollections.

We collect here those concepts from [Zelev] that are necessary to define the structure associated to the families of finite groups that we shall study. Apart from these we shall also frequently need notions and facts more specifically related to P.S.H.-algebras, in particular to the algebra $R(q)$ associated to the finite linear groups, but for those we shall confine ourselves to referring directly to [loc. cit.. We have retained the original notations wherever this would not prove to be rather impractical in our context.

2.1. *T-groups.*

We recall the notion of a *T-group* (trivialised group) from [Zelev 1.2]. A T-group A is a free Abelian group with a distinguished basis $\Omega(A)$; we call A the T-group generated by $\Omega(A)$. The concept of T-group is introduced to model the properties of the set of (virtual) characters of a finite group: for a finite group G we define the T-group $T(G)$ associated with G to be the T-group generated by the set of (characters of) irreducible finite dimensional complex representations of G. (Alternatively, this T-group may be described as the Grothendieck group of the category G-**Repr** of finite dimensional complex representations of G.) For general T-groups we use a terminology similar to that used for T-groups associated with a finite group, so $\Omega(A)$ will be called the set of *irreducible* elements of A. We consider $\mathbf{Z}$ as a T-group with $\Omega(\mathbf{Z}) = \{1\}$; it is the T-group associated with the trivial group.

For a T-group A we denote by A^+ the submonoid generated by $\Omega(A)$ (i.e., the set of elements whose coefficients with respect to the basis $\Omega(A)$ are all non-negative). The elements of A^+ are called *positive* (note that in particular we call $0 \in A$ positive) and we define a partial ordering on A by $a \leq b \iff b - a \in A^+$. A morphism of Abelian groups $f\colon A \to B$ between two T-groups is a T-group morphism if it is positive, i.e., if $f[A^+] \subseteq B^+$ (or equivalently, if it is monotonous, i.e., a morphism of partially ordered sets). In the T-group associated to a finite group, the positive elements are the ordinary (as opposed to virtual) characters. Consequently any additive functor G-**Repr** $\to$ H-**Repr** (for finite groups G, H) induces a T-group morphism $T(G) \to T(H)$. For $a \in A^+$ the fact that its coefficient of $\omega \in \Omega(A)$ is non-zero can be expressed by $\omega \leq a$; in this case we call ω a *constituent* of a.

There is a unique $\mathbf{Z}$-bilinear form on A with respect to which the basis $\Omega(A)$ is orthonormal; we call this the inner product on A, denoted as $\langle\cdot,\cdot\rangle$, and shall use geometric terminology, such as $a \perp b$ for $\langle a, b\rangle = 0$. For a T-group associated to a finite group, this is just the usual inner product of characters. The (irreducible) constituents ω of $a \in A^+$ can be alternatively characterised by $\langle \omega, a\rangle > 0$, and $\langle \omega, a\rangle$ is called the *multiplicity* of ω in a. We shall frequently use for $a, b \in A^+$ that $\langle a, b\rangle \geq 0$, and that any common constituent of a and b gives a positive contribution to $\langle a, b\rangle$; in particular if $\langle a, b\rangle = 1$ then a and b have a unique common constituent, and its multiplicity is 1 in both a and b.

In the category of T-groups we have the operations of direct sum of arbitrary families and tensor products of finite families. These are obtained from the similar constructions in the category of Abelian groups by defining $\Omega\bigl(\bigoplus_{i\in I} A_i\bigr)$ to be (the canonical image of) the disjoint union $\coprod_{i\in I}\Omega(A_i)$, and $\Omega\bigl(\bigotimes_{i=1}^n A_i\bigr)$ to be the canonical image of the cartesian product $\prod_{i=1}^n \Omega(A_i)$. For finite groups G, H the forming of tensor products of representations defines a bifunctor G-**Repr**$\times H$-**Repr** $\to (G\times H)$-**Repr**, additive in both arguments; this gives a T-group morphism $T(G) \otimes T(H) \to T(G \times H)$ which is known to be an isomorphism. We shall therefore identify $T(G \times H) = T(G) \otimes T(H)$.

A grading on a T-group A with values in a given commutative monoid M is a map $\deg\colon \Omega(A) \to M$. Linear combinations of irreducible elements with the same image $m \in M$ are called homogeneous elements of degree m (note that irreducible elements are by definition homogeneous). If a T-group B also has a grading with values in M, then a grading on $A \otimes B$ is induced by defining $\deg(a \otimes b) = \deg a + \deg b$. Also, a $\mathbf{Z}$-linear map $A \to B$ is said to preserve grading if sends homogeneous elements to homogeneous elements of the same degree. We shall reserve the term *graded T-group* for a T-group provided with a grading with natural numbers; a morphism of graded T-groups is a T-group morphism preserving this grading. A graded T-group A is usually given as $A = \bigoplus_{n\geq 0} A_n$ for some family $\{\, A_n \mid n \in \mathbf{N} \,\}$ of T-groups, in which case A_n is the set of homogeneous elements of degree n.

2.2. *The functors* i_a *and* r_a*.*

As mentioned above, additive functors between the categories of representations of finite groups give rise to morphisms between the associated T-groups. Fundamental examples of such functors are those of (co-)induction and restriction, to be discussed presently, and which themselves depend (in a functorial way) on group morphisms. Generalisations of these functors called i_a and r_a will be defined below (slightly different from the generalisations defined in [Zelev 8.1]), of which particular instances will be studied in detail.

We recall the definitions and properties of induction and restriction. Given a group morphism $f: H \to G$, any $\rho \in G$-**Repr** can be considered as H-representation by defining $h \in H$ to act as $f(h) \in G$. This H-representation is called the *restriction* $\operatorname{res}_f \rho$ of ρ via f (although the name seems less appropriate when f is not injective); obviously res_f is an additive functor. This functor has a left-adjoint ind_f, called *induction*, where $\operatorname{ind}_f \sigma$ is defined for $\sigma \in H$-**Repr** as the representation of G via action on the left factor on the quotient of $\mathbf{C}[G] \otimes_{\mathbf{C}} \sigma$ by the relations $gf(h) \otimes s = g \otimes (h \cdot s)$ for all $g \in G$, $h \in H$ and $s \in \sigma$. The adjointness of these functors is reflected by the fact that the associated maps on characters (also denoted ind_f and res_f) are adjoint with respect to the respective inner products: $\langle \operatorname{ind}_f \chi, \psi \rangle_G = \langle \chi, \operatorname{res}_f \psi \rangle_H$ for any H-character χ and G-character ψ; this fact (for injective f) is called the Frobenius reciprocity.

There is also a right-adjoint functor for res_f, which we call *coinduction*; $\operatorname{coind}_f \sigma$ is defined as the usual G-representation on the vector space of maps $\alpha: G \to \sigma$ that satisfy $\alpha(f(h)g) = h \cdot \alpha(g)$ for all $g \in G$ and $h \in H$. Since both ind_f and coind_f as maps on characters are adjoint to res_f, we see that any representation $\operatorname{coind}_f \sigma$ must be equivalent to $\operatorname{ind}_f \sigma$. In fact there is a natural equivalence of functors $\operatorname{coind}_f \dot{\to} \operatorname{ind}_f$ that sends the map $\alpha: G \to \sigma$ to the image of

$$\frac{1}{\# H} \sum_{g \in G} g \otimes \alpha(g^{-1}). \tag{1}$$

Consequently ind_f and coind_f may be used interchangeably. The reason for introducing coind_f is that it takes a particularly simple form when f is surjective: one easily shows that in that case $\operatorname{coind}_f \sigma$ is canonically isomorphic to the representation of $G \cong H / \operatorname{Ker} f$ on the subrepresentation of σ of vectors stable under $\operatorname{Ker} f$.

Now consider pairs $a = (p, q)$ of group morphisms $G \overset{p}{\leftarrow} H \overset{q}{\hookrightarrow} K$, where G, K are finite groups, H is a subgroup of K embedded by q, and p is surjective. Such pairs may be composed as follows: let $b = (r, s)$ be another such pair $K \overset{r}{\leftarrow} L \overset{s}{\hookrightarrow} M$, then define $N = r^{-1}[H] \subseteq L$, let $u: N \hookrightarrow M$ be its embedding, and define $t: N \to G$ by $t = p \circ r$, which is well defined and surjective. Now we define $b \circ a$ to be the pair (t, u). By direct calculation one shows that this composition is associative (so we may take these pairs to be the arrows in a category whose objects are finite groups). With a we associate functors $\mathrm{r}_a: K$-**Repr** $\to G$-**Repr** and $\mathrm{i}_a: G$-**Repr** $\to K$-**Repr** defined by

$$\mathrm{r}_a = \operatorname{coind}_p \circ \operatorname{res}_q \qquad \text{and} \qquad \mathrm{i}_a = \operatorname{ind}_q \circ \operatorname{res}_p . \tag{2}$$

Explicitly, for some $\rho \in K$-**Repr** we obtain $\mathrm{r}_a \rho$ by restricting ρ to the subgroup H, and then taking the subrepresentation fixed under $\operatorname{Ker} p$; the functor i_a is a left-adjoint for r_a, and $\mathrm{i}_a \sigma$ is obtained from $\sigma \in G$-**Repr** by 'lifting' to H (with $\operatorname{Ker} f$ acting trivially) and then inducing via q from H to K. It follows from the definitions that $\mathrm{r}_{b \circ a} = \mathrm{r}_a \circ \mathrm{r}_b$ (more formally: these functors are canonically naturally equivalent), and from this we get by adjunction that $\mathrm{i}_{b \circ a} = \mathrm{i}_b \circ \mathrm{i}_a$. So i_a and r_a depend in a 'functorial' way on a.

2.3. *A generalisation of Mackey's formula.*

Recall that for subgroups H, K of a finite group G we have Mackey's formula

$$\operatorname{res}_{K \hookrightarrow G} \circ \operatorname{ind}_{H \hookrightarrow G} = \bigoplus_{w \in K \backslash G / H} \operatorname{ind}_{i_w} \circ \operatorname{res}_{H \cap K^w \hookrightarrow H}, \tag{3}$$

where $K^w = w^{-1} K w$ and $i_w: H \cap K^w \to K$ is the injection given by $h \mapsto w h w^{-1}$. This formula should be interpreted as a natural equivalence of functors for any choice of representatives w for $K \backslash G / H$.

We shall generalise this formula to an expression for the composition $\mathrm{r}_b \circ \mathrm{i}_a$ for a, b suitable pairs of group morphisms.

Let pairs $a = (p, q): L \overset{p}{\leftarrow} H \overset{q}{\hookrightarrow} G$ and $b = (r, s): M \overset{r}{\leftarrow} K \overset{s}{\hookrightarrow} G$ be given with p and r surjective. Let $w \in G$ be a representative of some double coset in $K \backslash G / H$; as in Mackey's formula we consider the injections $H \cap K^w \hookrightarrow H$ and $i_w: H \cap K^w \to K$. Define $K_w \subseteq L$ to be the image of the composite map $H \cap K^w \hookrightarrow H \overset{p}{\to} L$, and $H_w \subseteq M$ to be the image of the composite map $H \cap K^w \overset{i_w}{\to} K \overset{r}{\to} M$. These composite maps define surjections of $H \cap K^w$ onto K_w and H_w of which we form the pushout: let Q_w be the quotient of $H \cap K^w$ by the (normal) subgroup generated by the kernels of these surjections. We obtain a diagram

$$\begin{array}{ccccc} L & \xleftarrow{\;p\;} & H & \xrightarrow{\;q\;} & G \\ \uparrow{\scriptstyle s_w} & & \uparrow & & \uparrow{\scriptstyle s} \\ K_w & \longleftarrow & H \cap K^w & \xrightarrow{\;i_w\;} & K \\ \downarrow{\scriptstyle r_w} & & \downarrow & & \downarrow{\scriptstyle r} \\ Q_w & \xleftarrow{\;p_w\;} & H_w & \xrightarrow{\;q_w\;} & M \end{array} \tag{4}$$

whose upper right-hand square need not commute, and where right and upward arrows are injections and the remainder surjections. Put $a_w = (p_w, q_w)$ and $b_w = (r_w, s_w)$.

2.3.1. Theorem. *There is a natural equivalence of functors* $\mathrm{r}_b \circ \mathrm{i}_a = \bigoplus_{w \in K \backslash G/H} \mathrm{i}_{a_w} \circ \mathrm{r}_{b_w}$.

Proof. From (4) we obtain a diagram of functors by replacing groups X by X-**Repr**, leftward and upward morphisms f by res_f in the opposite direction, rightward morphisms f by ind_f, and downward morphisms f by coind_f. The theorem then states that the composition of functors along the top and right edges is naturally equivalent to the sum over $w \in K\backslash G/H$ of the functors along the left and bottom edges; we check the 'commutativity' (i.e., natural equivalence of functors) of each of the squares separately, in each case summing over w those functors that depend on it (i.e., those not along the top or right edge). The commutativity of the upper right-hand square is given by Mackey's formula (3). In the upper left-hand square we only have restriction functors while the corresponding square of (4) commutes, so commutation follows for each w since res_f depends functorially on f. In the lower right-hand square we have functors of induction horizontally and functors of coinduction vertically, and by the natural equivalence of coind_f with ind_f we can proceed as for the previous square. Finally, for the lower left-hand square we have functors of restriction and coinduction while the corresponding square of (4) is a pushout of surjective group morphisms; for this case one easily proves commutativity directly from the descriptions of the functors. □

Remark. We use this theorem in place of [Zelev A3.1], which we find slightly less transparent.

§3. The T-groups associated to finite classical groups.

In this section we study certain graded T-groups that are associated to a family of finite classical groups. We proceed in close analogy to [Zelev §9] which treats the case of the general linear groups; we find in fact a structure that relates the groups we consider to the structure defined for that case.

3.1. *The groups G_n and their associated T-groups.*

We shall consider certain families $\{ G_n \mid n \in \mathbf{N} \}$ of classical groups defined over a finite field $\mathbf{F}_q$; the several possible choices for this family will be discussed presently. For such a family we shall study the graded T-group $A = \bigoplus_{n\geq 0} A_n$ where $A_n = T(G_n)$, which we shall endow with a module and comodule structure over the P.S.H.-algebra $R(q)$ of [Zelev §9]. We write $R = R(q)$ and recall its definition, rephrased in our terminology; the definitions for A will be modeled after those for R.

As graded T-group we have $R = \bigoplus_{n\geq 0} R_n$ where $R_n = T(GL_n(\mathbf{F}_q))$. Its P.S.H.-algebra structure is obtained by identifying R_0 with $\mathbf{Z}$ and defining morphisms $\nabla: R \otimes R \to R$ and $\Delta: R \to R \otimes R$ of multiplication and comultiplication as follows (these morphisms are called m and m^* respectively in [Zelev]). For $i, j \in \mathbf{N}$ let $P_{i,j}$ be the stabiliser subgroup in $GL_{i+j}(\mathbf{F}_q)$ of $\mathbf{F}_q^i$ viewed as subspace of $\mathbf{F}_q^{i+j}$—if $i, j > 0$ it is a maximal parabolic subgroup for the obvious BN-pair—and let $a_{i,j} = (p, q)$ be the pair of morphisms where $p: P_{i,j} \to GL_i(\mathbf{F}_q) \times GL_j(\mathbf{F}_q)$ is given by the action on the spaces $\mathbf{F}_q^i$ and $\mathbf{F}_q^{i+j}/\mathbf{F}_q^i$, and $q: P_{i,j} \hookrightarrow GL_{i+j}$ is the embedding. We define the adjoint functors $\mathrm{r}_{a_{i,j}}$ and $\mathrm{i}_{a_{i,j}}$ by (2), which induce adjoint morphisms $\Delta_{i,j}: R_{i+j} \to R_i \otimes R_j$ and $\nabla_{i,j}: R_i \otimes R_j \to R_{i+j}$. Summing over all $i + j = n$ we obtain morphisms between the homogeneous components R_n and $(R \otimes R)_n$, which combine to the pair of adjoint graded T-group morphisms Δ and ∇. The associative law $\nabla \circ (\nabla \otimes Id) = \nabla \circ (\mathrm{Id} \otimes \nabla)$, and the similar coassociative law, follow from the fact that the composite pairs of morphisms that define these maps on any component $R_i \otimes R_j \otimes R_k$ are equal; we denote this pair as $a_{i,j,k}$. Both morphisms of $a_{i,j,k}$ have as domain the stabiliser subgroup $P_{i,j+k} \cap P_{i+j,k}$ in $GL_{i+j+k}(\mathbf{F}_q)$ of the two-part flag $\mathbf{F}_q^i \subseteq \mathbf{F}_q^{i+j}$ of subspaces of $\mathbf{F}_q^{i+j+k}$.

The groups G_n will be defined in a geometric way, as groups of linear transformations of a vector space preserving some given 'metric' structure; for general facts about such groups we refer to [Artin], and to [Dieud] for a more comprehensive account. We fix here some geometric terminology. Consider a finite dimensional vector space V over $\mathbf{F}_q$ equipped with a form $f: V \times V \to \mathbf{F}_q$ such that either (i) f is bilinear and *alternate*, i.e., $f(x, x) = 0$ for all $x \in V$; or (ii) $\operatorname{char} \mathbf{F}_q \neq 2$ and f is bilinear and *symmetric*, i.e., $f(x, y) = f(y, x)$ for all $x, y \in V$; or (iii) q is a square, and f is sesquilinear and *hermitian* with respect to the unique automorphism $x \mapsto \overline{x} = x^{\sqrt{q}}$ of order 2 of $\mathbf{F}_q$, i.e., $f(\lambda x, \mu y) = \lambda\overline{\mu} f(x, y)$ and $f(y, x) = \overline{f(x, y)}$. If $f(x, y) = 0$ we shall call x and y *perpendicular*, and write $x \perp y$; in each case this is a symmetric relation. For subspaces X, Y we write $X \perp Y$ if f vanishes on $X \times Y$, and if $X \perp X$ we call X an *isotropic* subspace; a vector spanning an isotropic line is called an isotropic vector. We denote by $X^\perp$ the space of vectors $y \in V$ perpendicular to all $x \in X$, and by $\operatorname{rad} X$ the *radical* $X \cap X^\perp$ of X; we call X a *non-degenerate* subspace if $\operatorname{rad} X = 0$, which implies that $V = X \oplus X^\perp$. If V is non-degenerate then we call it a *symplectic space* if f is alternate, an *orthogonal space* if f is symmetric, and a *unitary space* if f is hermitian. A non-degenerate (sub-)space of dimension 2 that contains an isotropic line is called a *hyperbolic plane*. In any hyperbolic plane we can find a pair of isotropic vectors $x, y \in X$ such that $f(x, y) = 1$ (see [Artin Definition 3.8]; in particular any two hyperbolic planes are isomorphic) and conversely any such pair spans a hyperbolic plane; we call (x, y) a *hyperbolic pair*.

When the characteristic $\operatorname{char} \mathbf{F}_q$ is 2 we do not consider symmetric bilinear forms, as they do not lead to interesting groups; instead we consider spaces V equipped with a quadratic form $Q: V \to \mathbf{F}_q$, i.e., one that satisfies $Q(\lambda x) = \lambda^2 Q(x)$ and such that $f(x, y) = Q(x + y) + Q(x) + Q(y)$ defines a bilinear form. For a discussion of this case see [Dieud I §16]; when it applies we give a slightly different interpretation to the notions given above. Using the bilinear form f derived from Q—which is clearly alternate—we define the relation of being perpendicular as above, as well as the related notations. Rather than consider isotropic subspaces, however, we consider *singular* subspaces, i.e., subspaces on which Q vanishes (and which are therefore certainly isotropic for f); singular vectors are those that span singular lines. In this case $\operatorname{rad} X$ will denote the subspace of $X \cap X^\perp$ on which Q vanishes; the definition of non-degenerate subspaces can then be retained. If V is non-degenerate we call it an orthogonal space; note that in this case for any *singular* subspace X of V the codimension of $X^\perp$ equals $\dim X$. A hyperbolic plane is a non-degenerate 2-dimensional space containing a singular line; it will contain a pair (x, y) of singular vectors with $f(x, y) = 1$ (see [loc. cit.), which is called a hyperbolic pair.

3.1 The groups G_n and their associated T-groups

Along with specifying the family $\{G_n\}$ of groups we shall define for $i+j=n$ a subgroup $P'_{i,j}$ of G_n and a morphism $P'_{i,j} \to GL_i(\mathbf{F}_q) \times G_j$. Then the pair $a'_{i,j}$ of this morphism with the embedding $P'_{i,j} \hookrightarrow G_n$ defines functors that induce T-group morphisms $\nabla'_{i,j}: R_i \otimes A_j \to A_n$ and $\Delta'_{i,j}: A_n \to R_i \otimes A_j$. As in the case of GL_n these combine to an adjoint pair of graded T-group morphisms that we denote $\nabla_A: A \to R \otimes A$ and $\Delta_A: R \otimes A \to A$ and that will define the module and comodule structure of A over R.

Consider any symplectic, orthogonal or unitary space V over $\mathbf{F}_q$. If V contains an isotropic (resp. singular) line, then we can find another line not perpendicular to it, and these lines span a hyperbolic plane H; V then decomposes as an orthogonal direct sum $H \oplus H^\perp$. Now let V_0 be some symplectic, orthogonal or unitary space without isotropic (singular) lines, and for $n > 0$ construct spaces V_n from V_0 by taking the orthogonal direct sum with n hyperbolic planes. We choose in the ith hyperbolic plane a hyperbolic pair (e_i, e_{2n+1-i}), and define a flag $X_0 \subset X_1 \subset \cdots \subset X_n$ of isotropic (singular) subspaces of V_n by $X_i = \langle e_1, \ldots, e_i \rangle$. We have $\dim X_i^\perp = \dim V_0 + 2n - i$ and $\operatorname{rad} X_i^\perp = X_i$; the space $X_i^\perp / X_i$ has an induced structure of symplectic, orthogonal or unitary space, and is in fact isomorphic to V_{n-i}. We may now define the group G_n as the automorphism group of V_n, the subgroup $P'_{i,j}$ for $j = n-i$ as the stabiliser of X_i, and the morphism $P'_{i,j} \to GL_i(\mathbf{F}_q) \times G_j$ by its action on X_i and $X_i^\perp / X_i$. Choosing a complementary subspace to X_i in $X_i^\perp$ we easily verify that the latter morphism is surjective.

We define subgroups $B, N \subseteq G_n$, where B is the stabiliser of the flag $X_0 \subset \cdots \subset X_n$ and N is the stabiliser of the set of lines $\{ \langle e_j \rangle \mid 1 \le j \le 2n \}$. These satisfy the requirements for a BN-pair in G_n, as given in [Tits 3.2.1], except possibly the last one ($BN\ 2''$) which fails if V_1 has exaclty 2 isotropic lines, i.e., if $V_0 = 0$ in the orthogonal case. When (B, N) does form a BN-pair the groups $P'_{i,j}$ are parabolic subgroups with respect to this pair since clearly $P'_{i,j} \supseteq B$; in fact they are maximal parabolic subgroups, with the exception of $P'_{0,n} = G_n$ which is too big. Informally speaking we may say that ∇_A and Δ_A represent the operations of 'parabolic' induction and restriction, in analogy of ∇ and Δ.

Clearly the family $\{G_n\}$ constructed above and the additional data are completely determined by the type of metric and the choice of the space V_0. In the case of symplectic spaces we can only have $V_0 = 0$, so we obtain the family $G_n = Sp_{2n}(\mathbf{F}_q)$ of symplectic groups. Unitary spaces exist only if q is a square. If so, any element of the subfield $\mathbf{F}_{\sqrt{q}}$ fixed by the involutive automorphism of $\mathbf{F}_q$ can be written as a norm $x\overline{x}$ for some $x \in \mathbf{F}_q$, and it follows that any unitary space has an orthonormal base (see [Dieud I §8]), and equal dimensional spaces are equivalent. In particular, any unitary space of dimension 2 is a hyperbolic plane, so $\dim V_0$ is either 0 or 1. Both cases give rise to a family of groups: $G_n = U_{2n}(\mathbf{F}_q)$ respectively $G_n = U_{2n+1}(\mathbf{F}_q)$. The case of orthogonal spaces is the most complicated. We use the fact that, since multiplication of the form f or Q by a non-zero scalar transforms hyperbolic planes into other hyperbolic (hence isomorphic) planes, such multiplication of the form on V_0 will have no effect, up to isomorphism, on the groups. Because $\mathbf{F}_q$ is finite we have that $\dim V_0 \le 2$ (see [Artin III 6] and [Dieud I §16]), and there are three essentially different cases: (i) $V_0 = 0$; (ii) $V_0 = \mathbf{F}_q$ with $f(x,y) = xy$ respectively $Q(x) = x^2$ if $\operatorname{char} \mathbf{F}_q = 2$; (iii) $V_0 = \mathbf{F}_q^2$ with $f(x,y) = x_1 y_1 - g x_2 y_2$ where $g \in \mathbf{F}_q$ is a non-square, respectively $Q(x) = g x_1^2 + x_1 x_2 + g x_2^2$ where $g \in \mathbf{F}_q$ is such that $gX^2 + X + g$ is irreducible in $\mathbf{F}_q[X]$. Consequently we need to specify the form f or Q in even dimensions: we have three families: $G_n = O_{2n}(\mathbf{F}_q, f)$ or $O_{2n}(\mathbf{F}_q, Q)$, respectively $G_n = O_{2n+1}(\mathbf{F}_q)$ and $G_n = O_{2n+2}(\mathbf{F}_q, f)$ or $O_{2n+2}(\mathbf{F}_q, Q)$, where the forms f or Q are of the appropriate type.

The case $G_n = O_{2n+1}(\mathbf{F}_q)$ with $\operatorname{char} \mathbf{F}_q = 2$ is somewhat special, as we have $\dim V_n^\perp = 1$ (this is called the *defect* of Q), and any automorphism of the *symplectic* space $V_n / V_n^\perp$ is induced by a unique automorphism of the orthogonal space V_n (see [Dieud I §16]), so we have $G_n \cong Sp_{2n}(\mathbf{F}_q)$. Moreover the subset of V_n defined by $Q(x) = 0$ is naturally in bijection with $V_n / V_n^\perp$, and under this bijection the singular subspaces of V_n correspond precisely to the isotropic subspaces of $V_n / V_n^\perp$; therefore the subgroups $P'_{i,j}$ of G_n also correspond to those given for $Sp_{2n}(\mathbf{F}_q)$, and this case is completely equivalent to the symplectic case so it need not be separately considered. Hence we may assume if $\operatorname{char} \mathbf{F}_q = 2$ that any orthogonal space considered has even dimension, and consequently that the defect of Q is zero. For that case, as well as for orthogonal spaces with $\operatorname{char} \mathbf{F}_q \ne 2$ and arbitrary symplectic and unitary spaces, we have *Witt's theorem*: any isomorphism between a pair of subspaces may be extended to the entire space (see [Dieud I §11, §16]). One consequence of this theorem is that G_n acts transitively on the set of maximal isotropic (singular) flags in V_n, in particular the largest part of such a flag always has dimension $\dim X_n = n$.

This completes the choices of families $\{G_n\}$ that we shall consider; as stated above, the given data define for each choice a graded T-group $A = \bigoplus_{n \ge 0} T(G_n)$ and an adjoint pair of graded morphisms $\nabla_A: A \to R \otimes A$ and $\Delta_A: R \otimes A \to A$. We now determine the fundamental properties of these morphisms, which will be used later to determine the structure of A in an axiomatic way. First we show that they satisfy the appropriate (co-)associative laws, which will justify the statement that they define on A a module and comodule structure over R.

Consider the composition $(\mathrm{Id}_R \otimes \Delta_A) \circ \Delta_A$, which decomposes as sum of morphisms $(\mathrm{Id}_{R_i} \otimes \Delta'_{j,k}) \circ \Delta'_{i,j+k}$ from A_n to $R_i \otimes R_j \otimes A_k$ for $i+j+k=n$. By definition this morphism is induced by the functor r_a, where a is the result of composing the pair $a'_{i,j+k}$ of morphisms with the pair $\mathrm{Id}_{GL_i} \times a'_{j,k}$; the morphisms of a have as domain the stabiliser subgroup $P'_{i,j+k} \cap P'_{i+j,k}$ of the two-part isotropic (singular) flag $X_i \subseteq X_{i+j}$ in V_n. The same pair of morphisms is found if we compose $a'_{i+j,k}$ with $a_{i,j} \times \mathrm{Id}_{G_k}$; therefore we obtain the coassociative law $(\mathrm{Id}_R \otimes \Delta_A) \circ \Delta_A = (\Delta \otimes \mathrm{Id}_A) \circ \Delta_A$. Since also $\Delta'_{0,n} \colon A_n \to R_0 \otimes A_n$ is given by $a \mapsto 1 \otimes a$, it follows that Δ_A makes A into a comodule over the coalgebra R. Passing to adjoints we obtain the associative law $\nabla_A(r \otimes \nabla_A(s \otimes a)) = \nabla_A(\nabla(r \otimes s) \otimes a)$ for $r, s \in R$ and $a \in A$, and the multiplication ∇_A makes A into a module over the algebra R.

At this point we introduce some notation relating both to ∇ and Δ and to ∇_A and Δ_A. As usual we contract the multiplication $\nabla(r \otimes s)$ to $r.s$ $(r, s \in R)$ and similarly we write $\nabla_A(r \otimes a) = r.a$ for $r \in R$ and $a \in A$. For Δ we adopt from [Sweed 1.2] the following convention called 'sigma-notation', which is quite convenient, although it is not used in [Zelev]: the symbol '$\sum_{(r)}$' is defined such that

$$\Delta r = \sum_{(r)} r_1 \otimes r_2 \tag{5}$$

holds, meaning that r_1 and r_2 range over an appropriate set of values. Such values are not unique, but nevertheless expressions of the form $\sum_{(r)} E(r_1, r_2)$ will be well defined if E is a ($\mathbf{Z}$-)bilinear: such an E induces a linear map E' on $R \otimes R$, and the given expression then denotes the value $E'(\Delta r)$. Since comultiplication in R is coassociative, we may extend this notation to more tensorands, such that we have $\sum_{(r)} \Delta r_1 \otimes r_2 = \sum_{(r)} r_1 \otimes \Delta r_2 = \sum_{(r)} r_1 \otimes r_2 \otimes r_3$. Similarly the symbol '$\sum_{(a)}$' is defined for $a \in A$, such that $\Delta_A a = \sum_{(a)} a_1 \otimes a_0$ holds, and $\sum_{(a)} \Delta a_1 \otimes a_0 = \sum_{(a)} a_1 \otimes \Delta_A a_0 = \sum_{(a)} a_1 \otimes a_2 \otimes a_0$ (note that the tensorand in A always gets subscript '0'; the order of the remaining subscripts is arbitrary since comultiplication in R is cocommutative).

3.2. *Computation of* $\Delta_A \circ \nabla_A$.

A last fundamental property of Δ_A and ∇_A is obtained by applying 2.3.1 to compute the composition $\Delta_A \circ \nabla_A$. It is in this computation that the more specific properties of the groups G_n play a rôle. So consider a particular composition $\Delta'_{k,l} \circ \nabla'_{i,j}$ of components of Δ_A and ∇_A (with $i+j=k+l=n$). We first have to compute the set $P'_{k,l} \setminus G_n / P'_{i,j}$ of double cosets. We have noted that in several cases there is a BN-pair in G_n for which the $P'_{i,j}$ are parabolic subgroups; it is then known that these double cosets correspond bijectively to certain double cosets of the Weyl group of this BN-pair, which is isomorphic to the nth hyperoctahedral group H_n. We shall show that in all cases $P'_{k,l} \setminus G_n / P'_{i,j}$ is in bijection with this same set of double cosets of H_n, meanwhile giving an explicit parametrisation of these double cosets.

Since G_n acts transitively on the set of isotropic (resp. singular) subspaces of dimension i of V_n, and $P'_{i,j}$ is the stabiliser of such a subspace, it follows that $P'_{k,l} \setminus G_n / P'_{i,j}$ is in bijection with the set of orbits for the diagonal G_n-action on the set of pairs (X, Y) of isotropic (singular) subspaces of V_n of dimensions i and k respectively. Here the double coset represented by some $g \in G_n$ corresponds to the orbit of (gX_i, X_k). Now for the orbit of such a pair (X, Y) the following 3×3-matrix of natural numbers is well defined: its entries are the dimensions of the intersections of each of the vector spaces X, $X^\perp/X$, and $V_n/X^\perp$ with each of the vector spaces Y, $Y^\perp/Y$ and $V_n/Y^\perp$, where we define $A \cap (B/C)$ as $(A \cap B)/(A \cap C)$ and $(A/B) \cap (C/D)$ as $(A \cap C)/\big((A \cap D) + (B \cap C)\big)$. The numbers in the first row are the dimensions of intersections with the subspace X, whence their sum must be $i = \dim X$; similarly the sums of the numbers in each row and column are predetermined: they are $i, \dim V_n - 2i, i$ for the rows and $k, \dim V_n - 2k, k$ for the columns. Furthermore the entries in diametrically opposite positions must be equal, since the bilinear form induces a perfect pairing of the vector spaces corresponding to these positions.

We shall now show that the correspondence of orbits of pairs (X, Y) with 3×3-matrices satisfying the given constraints is bijective. First we show that pairs (X, Y) and (X', Y') giving rise to the same matrix lie in the same G_n-orbit. It will be sufficient to establish an isomorphism $X + Y \xrightarrow{\sim} X' + Y'$ that preserves the form f (resp. Q) and sends X to X' and Y to Y'; by Witt's theorem this isomorphism may then be extended to V_n, proving that (X, Y) and (X', Y') lie in the same G_n-orbit. The condition $X \perp Y$

is equivalent to $\dim\bigl(X \cap (V_n/Y^\perp)\bigr) = 0$, and therefore to $X' \perp Y'$; if it holds then $X + Y$ is isotropic (singular), and since $\dim(X \cap Y) = \dim(X' \cap Y')$ is also given the required isomorphism $X + Y \xrightarrow{\sim} X' + Y'$ is readily constructed. Now assume that $X \not\perp Y$, then there exist $x \in X, y \in Y$ with $f(x, y) = 1$, so that (x, y) is a hyperbolic pair; similarly there exists a hyperbolic pair (x', y') in $X' + Y'$. A direct computation shows that the matrix corresponding to the pair $(X \cap \langle y\rangle^\perp, Y \cap \langle x\rangle^\perp)$ is completely determined by the original one (the entries at positions $(1, 3)$ and $(3, 1)$ are decreased by 1) and hence it equals the matrix corresponding to $(X' \cap \langle y'\rangle^\perp, Y' \cap \langle x'\rangle^\perp)$. We may therefore by induction on $\dim(X + Y)$ assume the existence of an appriopriate isomorphism $(X \cap \langle y\rangle^\perp) + (Y \cap \langle x\rangle^\perp) \xrightarrow{\sim} (X' \cap \langle y'\rangle^\perp) + (Y' \cap \langle x'\rangle^\perp)$; extending it by the obvious isomorphism of hyperbolic planes $\langle x, y\rangle \xrightarrow{\sim} \langle x', y'\rangle$ we obtain the desired isomorphism $X + Y \xrightarrow{\sim} X' + Y'$.

Next we show that there is an orbit of pairs (X, Y) corresponding to any matrix satisfying the given conditions, meanwhile also establishing the correspondence with double cosets of H_n. Now for fixed n, i, k the conditions on the matrices differ between the various choices for $\{G_n\}$, depending on $\dim V_n$. We eliminate this difference by decreasing the middle entry of the matrix by $\dim V_0$ (the corresponding space has a summand isomorphic to V_0, so the remaining value will be non-negative). Thus the appropriate matrices for n, i, k are those with row sums $i, 2j, i$, column sums $k, 2l, k$ (with $i + j = k + l = n$), and equal entries at diametrically opposite positions.

Recall that N is the stabiliser in G_n of the set of lines $\{\, \langle e_m\rangle \mid 1 \le m \le 2n \,\}$; its action on that set defines a group morphism $N \to S_{2n}$. The image of this morphism can be seen to be the centraliser of the involution $(2n, 2n-1, \ldots, 2, 1) \in S_{2n}$, and it may be identified with H_n. For $m \le n$ let $L_m \subseteq H_n$ be the stabiliser of the subset $\{1, \ldots, m\}$; we consider the double cosets in $L_k \setminus H_n/L_i$. The double coset of $w \in H_n$ is characterised by the 3×3-matrix of cardinalities of the intersections of each of the sets $w\{1, \ldots, i\}$, $w\{i + 1, \ldots, 2n - i\}$, and $w\{2n - i + 1, \ldots, 2n\}$, with each of the sets $\{1, \ldots, k\}$, $\{k + 1, \ldots, 2n - k\}$, and $\{2n - k + 1, \ldots, 2n\}$, and this matrix is appropriate for n, i, k. It follows directly from the definitions that for $g \in N$ the matrix characterising the double coset of its image in H_n is the same as that corresponding to the orbit of the pair $(X, Y) = (gX_i, X_k)$ (after decreasing the middle entry by $\dim V_0$). Conversely, given such a matrix $B = (b_{i,j})_{1\le i,j\le 3}$ appropriate for n, i, k we may partition the set $\{1, \ldots, 2n\}$ into consecutive blocks of sizes $b_{1,1}, b_{1,2}, b_{1,3}, b_{2,1}, b_{2,2}, b_{2,3}, b_{3,1}, b_{3,2}, b_{3,3}$ and take w to be the permutation that rearranges these blocks in the order $b_{1,1}, b_{2,1}, b_{3,1}, b_{1,2}, b_{2,2}, b_{3,2}, b_{1,3}, b_{2,3}, b_{3,3}$; then $w \in H_n$ and its double coset is characterised by the matrix B. Thus we have shown that both $P'_{k,l} \setminus G_n/P'_{i,j}$ and $L_k \setminus H_n/L_i$ are in bijection with the set of matrices appropriate for n, i, k.

Now fix such a matrix

$$\begin{pmatrix} a & b & c \\ d & e & d \\ c & b & a \end{pmatrix},$$

choose a representative $w \in G_n$ of the corresponding double coset, and consider the diagram (4) for this situation: we have $G = G_n$, $L = GL_i \times G_j$, $M = GL_k \times G_l$, $H = P'_{i,j}$, and $K = P'_{k,l}$. It is not difficult to express the remaining groups and morphisms in terms of stabilisers of subspaces; e.g., $H \cap K^w$ is the simultaneous stabiliser in G_n of the subspaces X_i and $w^{-1}X_k$ of V_n, and $K_w \subseteq GL_i \times G_j$ is the simultaneous stabiliser of the flag $X_i \cap w^{-1}X_k \subseteq X_i \cap w^{-1}X_k^\perp$ in X_i and of the subspace $(X_i^\perp/X_i) \cap w^{-1}X_k$ of $X_i^\perp/X_i$. The quotient Q_w of K_w is isomorphic to $(GL_a \times GL_b \times GL_c) \times \bigl(GL_d \times G_{e/2}\bigr)$, where the consecutive factors correspond to the action of K_w on the spaces $X_i \cap w^{-1}X_k$, $X_i \cap w^{-1}(X_k^\perp/X_k)$, $X_i \cap w^{-1}(V_n/X_k^\perp)$, $(X_i^\perp/X_i) \cap w^{-1}X_k$, and $(X_i^\perp/X_i) \cap w^{-1}(X_k^\perp/X_k)$. Viewed as quotient of H_w the factors of Q_w correspond to the action of H_w on the w-images of these spaces, with one important exception: the factor GL_c corresponds to the action on $w(V_n/X_i^\perp) \cap X_k$ rather than on $wX_i \cap (V_n/X_k^\perp)$; these spaces are dual to each other.

It follows from these considerations that for this situation the composite functor $\mathrm{i}_{a_w} \circ \mathrm{r}_{b_w}$ appearing in 2.3.1 can be described as follows. First apply the functor $\mathrm{r}_{a_{a,b,c}} \otimes \mathrm{r}_{a'_{d,e/2}}\colon (GL_i \times G_j)$-**Repr** $\to \bigl((GL_a \times GL_b \times GL_c) \times \bigl(GL_d \times G_{e/2}\bigr)\bigr)$-**Repr**, (recall that the pair $a_{a,b,c}$ of morphisms comes from $P_{a,b+c} \cap P_{a+b,c} \subseteq GL_i$, and the pair $a'_{d,e/2}$ comes from $P'_{d,e/2} \subseteq G_j$). Then apply the functor from $\bigl((GL_a \times GL_b \times GL_c) \times \bigl(GL_d \times G_{e/2}\bigr)\bigr)$-**Repr** to $\bigl((GL_a \times GL_d \times GL_c) \times \bigl(GL_b \times G_{e/2}\bigr)\bigr)$-**Repr** that permutes the factors GL_b and GL_d, and applies the automorphism of taking transpose inverses to factor GL_c (this automorphism corresponds to the action of GL_c on the dual of the space $\mathbf{F}_q^c$ that it naturally acts on). Finally apply the functor $\mathrm{i}_{a_{a,d,c}} \otimes \mathrm{i}_{a'_{b,e/2}}$ to $(GL_k \times G_l)$-**Repr**.

Remark. These considerations can be straightforwardly generalised to the case of double cosets of G_n by stabilisers of arbitrary isotropic or singular flags: one then uses larger matrices, with constraints similar to those above. The results found in this way, however, also follow by repeated application of those above.

We interpret these matters in terms of Δ, ∇, Δ_A and ∇_A. It is clear that the functors $\mathrm{r}_{a_{a,b,c}}$ and $\mathrm{i}_{a_{a,d,c}}$ above correspond to components of the maps of repeated comultiplication $R \to R \otimes R \otimes R$ respectively repeated multiplication $R \otimes R \otimes R \to R$; we denote these maps as Δ^2 and ∇^2. For each c the involutive automorphism of GL_c of taking transpose inverses induces a nontrivial involution on R_c; this defines an involutive P.S.H.-algebra automorphism τ of R. Summing over all matrices appropriate for n, i, k precisely combines all components of Δ^2 and ∇^2 that contribute to the component of $\Delta_A \circ \nabla_A$ mapping $R_i \otimes A_j \to R_k \otimes A_l$. Therefore, we obtain the following statement about the morphisms Δ_A and ∇_A.

3.2.1. Theorem. *The morphisms Δ_A and ∇_A satisfy the relation*

$$\Delta_A \circ \nabla_A(r \otimes a) = \sum_{(r)} \sum_{(a)} r_1.a_1.\tau(r_2) \otimes r_3.a_0,$$

valid for all $r \in R$ and $a \in A$. □

As stated in [Zelev 9.3], the algebra R decomposes as tensor product $\bigotimes_{\rho\in\mathcal{C}} R(\rho)$ of elementary P.S.H.-subalgebras, where $\mathcal{C}$ is the set of so-called cuspidal representations, and each tensorand $R(\rho)$ has a unique non-trivial automorphism. With respect to this structure the following can be said about τ. It is shown in [Zelev 9.4,9.5] that a map $\delta\colon R \to \mathbf{Z}$ can be defined such that—for a chosen map $\delta_y\colon R_E \to \mathbf{Z}$ defined on the standard elementary P.S.H.-algebra R_E—there is for each ρ a *unique* isomorphism $R(\rho) \xrightarrow{\sim} R_E$ that commutes with the δ's. (Since there are two isomorphisms if we omit this commutation requirement, it is clear that the definition of δ involves more than just the P.S.H.-algebra structure; in fact, for $r \in R_n^+$ the number $\delta(r)$ is defined as the dimension of the weight space of the restriction of r to the subgroup U of unipotent upper triangular matrices in GL_n for a certain linear character $U \to \mathbf{C}^*$.) Calculation shows that $\delta \circ \tau = \delta$; therefore τ is completely determined by its effect on the set of cuspidal elements. In particular, if a cuspidal element ρ is fixed under τ, then so is every element of $R(\rho)$. We shall see that the latter fact also follows from facts already derived, since the assumption that τ acts as a non-trivial automorphism on any $R(\rho)$ would contradict the existence of non-zero modules A with morphisms Δ_A and ∇_A having the stated properties.

§4. Theory of twisted P.S.H.-modules.

In this section we axiomatise the properties found for the particular instances of T-groups A with morphisms ∇_A and Δ_A to the general concept of 'twisted P.S.H.-module'. From the axioms we then develop a theory for these objects, including theorems about composition and decomposition; we also describe two fundamental models for this theory in terms of the P.S.H.-algebra of [Zelev §3].

4.1. *Axioms and first results.*

We start with definitions, notations and some elementary results. Let a P.S.H.-algebra R be given, as defined in [Zelev 1.2]; we denote the morphisms of multiplication and comultiplication in R by ∇ and Δ respectively, and the counit map $R \to \mathbf{Z}$ by e^*, and the unit map $\mathbf{Z} \to R_0$ is used as an identification. Also let a graded T-group $A = \bigoplus_{i\geq 0} A_i$ be given, together with $\mathbf{Z}$-linear maps $\nabla_A\colon R \otimes A \to A$ and $\Delta_A\colon A \to R \otimes A$. As in our previous section we abbreviate $\nabla(r \otimes s) = r.s$ and $\Delta r = \sum_{(r)} r_1 \otimes r_2$, and similarly $\nabla_A(r \otimes a) = r.a$ and $\Delta_A a = \sum_{(a)} a_1 \otimes a_0$.

4.1.1. Definition. *The morphisms ∇_A and Δ_A give A the structure of a positive selfadjoint module over R (or P.S.-module for short) if the following conditions are satisfied for all $r, s \in R$, $a, b \in A$.*

1. *∇_A and Δ_A are morphisms of graded T-groups, i.e., they are positive and preserve grading;*
2. *A is a module over the algebra R via ∇_A, i.e., we have $(r.s).a = r.(s.a)$ and $1.a = a$;*
3. *A is a comodule over the coalgebra R via Δ_A, i.e., we have $\sum_{(a)} a_1 \otimes \Delta_A a_0 = \sum_{(a)} \Delta a_1 \otimes a_0$ and $\sum_{(a)} e^*(a_1)a_0 = a$;*
4. *∇_A and Δ_A are adjoint, i.e., we have $\langle r.a, b\rangle = \langle r \otimes a, \Delta_A b\rangle = \sum_{(b)} \langle r, b_1\rangle\langle a, b_0\rangle$.*

The last condition implies that conditions 2 and 3 are equivalent. By condition 2 we may write $r.s.a$ without parentheses, and similarly the initial expressions in condition 3 may be written as $\sum_{(a)} a_1 \otimes a_2 \otimes a_0$.

Morphisms of P.S.-modules are T-group morphisms that are morphisms of modules and of comodules over R as well.

We shall only consider P.S.-modules which in addition to these axioms satisfy an equation for the composition $\Delta_A \circ \nabla_A$. The most natural such equation is the Hopf axiom for modules/comodules over a Hopf algebra: $\Delta_A(r.a) = \sum_{(r),(a)} r_1.a_1 \otimes r_2.a_0$. P.S.-modules which satisfy this condition will be called P.S.H.-modules; the structure theory of such modules is quite trivial. However, in view of 3.2.1, our interest will be mainly in modules satisfying a "twisted" version of this Hopf axiom.

4.1.2. Definition. *Let τ be an involutive P.S.H.-algebra automorphism of R. A twisted P.S.H.-module over (R,τ) is a P.S.-module A over R such that $\Delta_A(r.a) = \sum_{(r),(a)} r_1.a_1.\tau(r_2) \otimes r_3.a_0$ for all $r \in R$ and $a \in A$.*

We can simplify this condition by defining a morphism

$$D = D_\tau = \nabla \circ (\mathrm{Id} \otimes \tau) \circ \Delta : R \to R. \tag{6}$$

Clearly this is a P.S.H.-algebra endomorphism, since both ∇ and Δ are P.S.H.-algebra morphisms, i.e., morphisms of T-groups, algebras, and coalgebras. Also, D is selfadjoint: $\langle D(r), s\rangle = \langle r, D(s)\rangle$. Using the commutativity of ∇ we can now rewrite the twisted Hopf axiom as

$$\Delta_A(r.a) = \sum_{(r),(a)} D(r_1).a_1 \otimes r_2.a_0\,. \tag{7}$$

We now introduce a number of concepts for P.S.-modules in analogy of [Zelev §1]. Recall that the counit $e^*\colon R \to \mathbf{Z}$ is the orthogonal projection onto $R_0 \cong \mathbf{Z}$; therefore by the second part of the comodule property we have for any $a \in A$ that $\Delta_A a \in 1 \otimes a + I \otimes A$, where $I = \bigoplus_{i>0} R_i = \operatorname{Ker} e^*$. Now we define $P_A = \{\, \alpha \in A \mid \Delta_A \alpha = 1 \otimes \alpha \,\}$; this is called the set of *primitive* elements of A. We have $a \in P_A$ if and only if $\Delta_A a \perp I \otimes A$, which by adjointness is equivalent to $a \perp IA$, where $IA = \nabla_A[I \otimes A]$. Hence the vector space $A \otimes \mathbf{Q}$ is the (orthogonal) direct sum of the subspaces $P_A \otimes \mathbf{Q}$ and $IA \otimes \mathbf{Q}$; this allows linear relations on A to be proved separately on P_A and IA. We write $\mathcal{C}_A = \Omega(A) \cap P_A$ for the set of irreducible primitive elements in A, which we shall henceforth call *cuspidal* elements. We shall show that any non-zero twisted P.S.H.-module A has at least one cuspidal element; if it has exactly one cuspidal element then we shall call A an *elementary* twisted P.S.H.-module.

We introduce operators adjoint to multiplication by fixed elements. Let $r, s, t \in R$ and $a, b \in A$. As in [Zelev] we define $r^*\colon R \to R$ by

$$r^*(s) = \sum_{(s)} \langle r, s_1\rangle s_2, \quad \text{so that by adjointness} \quad \langle r^*(s), t\rangle = \langle s, r.t\rangle. \tag{8a, b}$$

We similarly define $r^\star\colon A \to A$ by

$$r^\star(a) = \sum_{(a)} \langle r, a_1\rangle a_0, \quad \text{so that} \quad \langle r^\star(a), b\rangle = \langle a, r.b\rangle, \tag{9a, b}$$

and we define $a^*\colon A \to R$ by

$$a^*(b) = \sum_{(b)} \langle a, b_0\rangle b_1, \quad \text{so that} \quad \langle a^*(b), r\rangle = \langle b, r.a\rangle. \tag{10a, b}$$

Clearly, in all three cases, the second equation completely characterises the operator. Therefore, by the commutativity of multiplication, all operators r^* commute, as do all operators $r^\star$, and we have $r^* \circ a^* = a^* \circ r^\star$. Morphisms of modules over R can be characterised by their commutation with the multiplication by any $r \in R$; morphisms $A \to B$ of comodules over R can be similarly characterised as linear maps that commute with the operators $r^\star$ on A and B respectively, for any $r \in R$. In particular all maps $r^\star\colon A \to A$ and $a^*\colon A \to R$ are morphisms of comodules. As Δ and Δ_A are positive morphisms, it follows that for $r \in R^+$ and $a \in A^+$ the operators r^*, $r^\star$, and a^* are positive as well. Furthermore, for homogeneous elements, the corresponding operators have the inverse degree: if $r \in R_i$ and $a \in A_i$ then

$r^*[R_k] \subseteq R_{k-i}$, $r^\star[A_k] \subseteq A_{k-i}$ and $a^*[A_k] \subseteq R_{k-i}$, where the right hand sides are 0 when $k < i$. For pairs r, s or a, b that are homogeneous and of equal degree we have $r^*(s) = \langle r, s\rangle$ and $a^*(b) = \langle a, b\rangle$.

Now consider twisted P.S.H.-modules. Very useful forms of the (twisted) Hopf axiom are obtained in terms of the operators defined above. Note that by (7) for any bilinear expression E we may rewrite $\sum_{(r.a)} E\bigl((r.a)_1, (r.a)_0\bigr)$ as $\sum_{(r),(a)} E\bigl(D(r_1).a_1, r_2.a_0\bigr)$. Therefore we have, for instance,

$$\begin{aligned} r^\star(s.a) &= \sum_{(s.a)} \langle r, (s.a)_1\rangle (s.a)_0 = \sum_{(s),(a)} \langle r, D(s_1).a_1\rangle s_2.a_0 \\ &= \sum_{(r),(s),(a)} \langle r_1, D(s_1)\rangle \langle r_2, a_1\rangle s_2.a_0 \\ &= \sum_{(r),(s)} \langle D(r_1), s_1\rangle s_2.r_2^\star(a) \\ &= \sum_{(r)} D(r_1)^*(s).r_2^\star(a) \end{aligned}$$

In this way we find the three equations

$$r^*(s.t) = \sum_{(r)} r_1^*(s).r_2^*(t), \tag{11}$$

$$r^\star(s.a) = \sum_{(r)} D(r_1)^*(s).r_2^\star(a), \tag{12}$$

$$a^*(r.b) = \sum_{(a)} D\bigl(a_1^*(r)\bigr).a_0^*(b). \tag{13}$$

For primitive elements these equations simplify considerably. Recall from [Zelev 1.5] that the set P_R of primitive elements of R is defined as $\{\, r \in R \mid \Delta r = r \otimes 1 + 1 \otimes r \,\}$. Let $\rho \in P_R$ and $\alpha \in P_A$; then we have

$$\rho^*(r.s) = \rho^*(r).s + r.\rho^*(s), \tag{14}$$

$$\rho^\star(r.a) = (\rho + \tau(\rho))^*(r).a + r.\rho^\star(a), \tag{15}$$

$$\alpha^*(r.a) = D(r).\alpha^*(a); \tag{16}$$

also, for $r \in I$, the definitions give

$$r^*(\rho) = \langle r, \rho\rangle, \tag{17}$$

$$r^\star(\alpha) = 0, \tag{18}$$

$$a^*(\alpha) = \langle a, \alpha\rangle. \tag{19}$$

As a first application we prove the following

4.1.3. Proposition. *For all $r \in R$ and $a \in A$ we have $r.a = \tau(r).a$.*

Proof. As the relation is clearly linear in a, we may prove this separately for $a \in P_A$ and $a \in IA$.

a. Let $a \in P_A$. As τ is involutive we have $D(r) = D\bigl(\tau(r)\bigr)$, and by (16) we get $a^*(r.a) = D(r).a^*(a) = D\bigl(\tau(r)\bigr).a^*(a) = a^*\bigl(\tau(r).a\bigr)$. It follows that $\langle r.a, r.a\rangle = \langle \tau(r).a, r.a\rangle = \langle \tau(r).a, \tau(r).a\rangle$, so $r.a - \tau(r).a$ is perpendicular to both $r.a$ and $\tau(r).a$, whence it must be zero.

b. To prove the proposition for $a \in IA$ it suffices to consider $a = s.b$ with $s \in I$ and $b \in A$ homogeneous. Now $\deg b < \deg a$, so we may assume by induction that the proposition holds for any product $r.b$. Then we have $r.a = r.s.b = \tau(r.s).b = \tau(r).\tau(s).b = \tau(r).s.b = \tau(r).a$ □

4.2. *Fundamental examples, composition.*

We shall now show how examples of twisted P.S.H.-modules can be constructed. It is easy to construct untwisted P.S.H.-modules over any P.S.H.-algebra R; for instance, one may take $A = R$ with $\nabla_A = \nabla$ and $\Delta_A = \Delta$. With the following lemma we shall be able to derive twisted P.S.H.-modules from untwisted ones. The lemma deals with two different P.S.H.-algebras R and S, so we distinguish multiplication by elements of R, written '$r.x$', from that by elements of S, written '$s \times x$'; similarly $\sum'_{(x)}$ will be used to indicate comultiplication in S or from A to $S \otimes A$, as opposed to $\sum_{(x)}$ which is used in connection with R.

4.2.1. Lemma. *Let R and S be P.S.H.-algebras, $f: S \to R$ a P.S.H.-algebra morphism, and $f^*: R \to S$ the adjoint morphism, and let A be a P.S.-module over R that satisfies $\Delta_A(r.a) = \sum_{(r),(a)} g(r_1).a_1 \otimes r_2.a_0$ for all $r \in R$, $a \in A$, and some fixed $\mathbf{Z}$-linear map $g: R \to R$. Then A becomes a P.S.-module over S via the morphisms $\nabla'_A = \nabla_A \circ (f \otimes \mathrm{Id}_A)$ and $\Delta'_A = (f^* \otimes \mathrm{Id}_A) \circ \Delta_A$, which moreover satisfies*

$$\Delta'_A(s \times a) = \sum_{(s),(a)}{}' f^* \circ g \circ f(s_1) \times a_1 \otimes s_2 \times a_0$$

for $s \in S$ and $a \in A$.

Proof. The conditions in 4.1.1 are immediately verified, even without using that f is a coalgebra morphism. For the final statement we *do* need this hypothesis, as well as the dual statement that f^* is an algebra morphism; these statements can be expressed respectively as $\sum_{(f(s))} f(s)_1 \otimes f(s)_2 = \sum'_{(s)} f(s_1) \otimes f(s_2)$ and $f^*(r.r') = f^*(r) \times f^*(r')$. Also, we have by definition $s \times a = f(s).a$ and $\sum'_{(a)} a_1 \otimes a_0 = \sum_{(a)} f^*(a_1) \otimes a_0$. Therefore

$$\begin{aligned}
\Delta'_A(s \times a) = \Delta'_A(f(s).a) &= \sum_{(f(s).a)}{}' (f(s).a)_1 \otimes (f(s).a)_0 \\
&= \sum_{(f(s).a)} f^*\big((f(s).a)_1\big) \otimes (f(s).a)_0 \\
&= \sum_{(f(s)),(a)} f^*\big(g(f(s)_1).a_1\big) \otimes f(s)_2.a_0 \\
&= \sum_{(s)}{}' \sum_{(a)} f^* \circ g \circ f(s_1) \times f^*(a_1) \otimes f(s_2).a_0 \\
&= \sum_{(s),(a)}{}' f^* \circ g \circ f(s_1) \times a_1 \otimes s_2 \times a_0
\end{aligned}$$ □

This lemma may be applied in two ways—both with $g = \mathrm{Id}_R$—to obtain twisted P.S.H.-modules from untwisted ones. Let R_E be the P.S.H.-algebra with a single irreducible primitive element, whose structure is derived in [Zelev §§3,4]; there it is called the 'universal' P.S.H.-algebra, but we prefer to call it elementary, in agreement with our similar terminology for modules. We denote its multiplication and comultiplication by ∇_{R_E} and Δ_{R_E} respectively. We may now first take $S = R_E$, $R = R_E \otimes R_E$ and $f = \Delta_{R_E}$. Then $f^* = \nabla_{R_E}$ and for $\tau = \mathrm{Id}_R$ we have $f^* \circ f = D_\tau$, so that by the lemma any P.S.H.-module over $R_E \otimes R_E$ becomes a twisted P.S.H.-module over (R_E, Id). On the other hand we may take $R = R_E$, $S = R_E \otimes R_E$ and $f = \nabla_{R_E}$. Then $f^* = \Delta_{R_E}$ and it follows from the Hopf axiom for R_E and its (co-)commutativity that $f^* \circ f = \Delta_{R_E} \circ \nabla_{R_E} = D_\chi$, where χ is given by $\chi(x \otimes y) = (y \otimes x)$. Therefore by the lemma any P.S.H.-module over R_E becomes a twisted P.S.H.-module over $(R_E \otimes R_E, \chi)$. We have in particular the following two fundamental models of the set of axioms of twisted P.S.H.-modules.

4.2.2. Proposition. *In the following two cases A is an elementary twisted P.S.H.-module over (R, τ):*

(I) $A = R_E$, $R = R_E \otimes R_E$ *and* $\tau = \chi$ *via the morphisms* $\nabla_A = \nabla_{R_E} \circ (\nabla_{R_E} \otimes \mathrm{Id}_A)$ *and* $\Delta_A = (\Delta_{R_E} \otimes \mathrm{Id}_A) \circ \Delta_{R_E}$.

(II) $A = R_E \otimes R_E$, $R = R_E$ *and* $\tau = \mathrm{Id}$ *via the morphisms* $\nabla_A = \nabla_{R_E \otimes R_E} \circ (\Delta_{R_E} \otimes \mathrm{Id}_A)$ *and* $\Delta_A = (\nabla_{R_E} \otimes \mathrm{Id}_A) \circ \Delta_{R_E \otimes R_E}$.

Proof. Both cases are derived as indicated above from the P.S.H.-algebras R_E respectively $R_E \otimes R_E$ as P.S.H.-module over itself. One easily verifies that the elements 1 respectively $1 \otimes 1$ are the sole cuspidal elements of A.

We now consider two ways in which twisted P.S.H.-modules may be composed into larger ones. For an arbitrary familiy $\{ A_i \mid i \in I \}$ of such modules over the same pair (R, τ) one can make $A = \bigoplus_{i \in I} A_i$ into a twisted P.S.H.-module over (R, τ) in an obvious way. The set $\mathcal{C}_A$ of cuspidals is in canonical bijection with $\coprod_i \mathcal{C}_{A_i}$ in this case. On the other hand, consider a family $\{ R_i \mid i \in I \}$ of P.S.H-algebras, each with an involution τ_i. Then $R = \bigotimes_i R_i$ is again a P.S.H.-algebra, and $\bigotimes_i \tau_i$ is an involution of R. Now for a finite family of twisted P.S.H.-modules A_i each over its own (R_i, τ_i) the tensor product $A = \bigotimes_i A_i$ can be made into a twisted P.S.H.-module over $(\bigotimes_i R_i, \bigotimes_i \tau_i)$ by defining ∇_A and Δ_A tensorand-wise. A similar construction can be performed for infinite families of modules, provided that we define $\bigotimes_i A_i$ to consist only of sums of tensors of which all but a finite number of tensorands are cuspidal of degree 0 (irreducibilty is needed to define $\Omega(A)$, and primitivity is necessary to ensure that no infinite sums arise in the definition of Δ_A). This construction is most useful when each A_i is elementary; then A is also elementary. In this case we can embed any tensor product of a finite subset of the factors into larger tensor products by inserting the unique cuspidal element into the vacant tensorand positions, and the infinite tensor product is the direct limit (in an appropriate category whose objects are triples (A, R, τ)) of this system of finite tensor products.

It will be the outcome of the structure theory of twisted P.S.H.-modules that, up to isomorphism, they can all be obtained from the two fundamental examples mentioned, by (apart from scaling/shifting the gradings) formation of tensor products and of direct sums (in that order), as described above. This will be proved by first showing that each twisted P.S.H.-module is a direct sum of elementary modules, then showing how these summands can be decomposed—under an additional assumption—as tensor products of twisted P.S.H.-modules over P.S.H.-subalgebras, and finally proving the uniqueness of elementary twisted P.S.H.-modules both over $(R_E \otimes R_E, \chi)$ and over (R_E, Id) (this is the hardest part), while showing *en passant* that the additional assumption is always satisfied.

4.3. *Decomposition.*

The first part of our analysis is very similar to [Zelev §2].

4.3.1. Lemma. *Let $r \in R^+$, $a \in A^+$ and $\omega, \omega' \in \Omega(A)$. If ω is a constituent of a, and ω' is a constituent of $r.\omega$, then ω' is also a constituent of $r.a$.*

Proof. Since ∇_A is monotonous we have $\omega' \le r.\omega \le r.a$. □

4.3.2. Proposition. *Every $a \in \Omega(A)$ is a constituent of some product $r.\alpha$, with $r \in R^+$ and $\alpha \in \mathcal{C}_A$.*

Proof. If $a \in P_A$ then a is cuspidal, so we may take $r = 1$ and $\alpha = a$. Otherwise we have $\Delta_A a - 1 \otimes a > 0$, whence it has a constituent $r \otimes b$ with $r \in I^+$ and $b \in \Omega(A)$, and since $\langle 1, r\rangle = 0$ we have $\langle \Delta_A a - 1 \otimes a, r \otimes b\rangle = \langle a, r.b\rangle > 0$. So a is a constituent of $r.b$, and since $\deg b < \deg a$ we may assume by induction that b is a constituent of $r'.\alpha$ for some $r' \in R^+$ and $\alpha \in \mathcal{C}_A$. Then by 4.3.1 a is a constituent of $r.r'.\alpha$. □

4.3.3. Theorem. *Any twisted P.S.H.-module A decomposes in a unique way as the direct sum*

$$A = \bigoplus_{\alpha \in \mathcal{C}_A} A(\alpha)$$

of elementary submodules $A(\alpha)$ of A.

Proof. For $\alpha \in \mathcal{C}_A$ define $\Omega(\alpha) \subseteq \Omega(A)$ to be the set of irrreducible elements appearing as a constituent of some $r.\alpha$ with $r \in R^+$, and define $A(\alpha)$ to be the $\mathbf{Z}$-span of $\Omega(\alpha)$. By the previous proposition we have $\Omega(A) = \bigcup_{\alpha \in \mathcal{C}_A} \Omega(\alpha)$, and the theorem will follow if the sets $\Omega(\alpha)$ are mutually disjoint. So let $\alpha, \alpha' \in \mathcal{C}_A$ and $r, r' \in R^+$, then we have by (16) and (19) that

$$\langle r.\alpha, r'.\alpha'\rangle = \langle r, \alpha^*(r'.\alpha')\rangle = \langle r, D(r').\alpha^*(\alpha')\rangle = \langle r, D(r')\rangle\langle \alpha, \alpha'\rangle \tag{20}$$

whence $r.\alpha$ and $r'.\alpha'$ can have a constituent in common only if $\alpha = \alpha'$, proving the required disjointness. □

According to this theorem the structure theory of twisted P.S.H.-modules reduces to the case of elementary modules; moreover, by a shift in the grading, we may assume that the unique cuspidal element has degree 0. From this point on we consider an elementary twisted P.S.H.-module A over (R, τ), with $\mathcal{C}_A = \{\alpha\}$ and $\deg \alpha = 0$.

According to [Zelev 2.5] the P.S.H.-algebra R can be graded by the free commutative monoid generated by $\mathcal{C}_R$—which we shall denote by $F(\mathcal{C}_R)$—in such a way that any constituent of $\prod_i \rho_i^{d_i}$ has degree $\sum_i d_i \rho_i$. We denote this grading (which is a refinement of the grading by natural numbers) by the symbol $\deg_{\mathcal{C}}$; it is preserved by the morphisms ∇ and Δ. We shall use this grading to put a similar grading on $A = A(\alpha)$, in such a way that the degree of $r.\alpha$ is determined by $\deg_{\mathcal{C}} r$. Note however that by 4.1.3 such a grading on A cannot reflect a distinction between ρ and $\tau(\rho)$ (when these actually differ). Therefore let $\mathcal{C}_R/\tau$ be the set of τ-orbits in $\mathcal{C}_R$ (each consisting of 1 or 2 elements), and let $\deg_{\mathcal{C}/\tau}$ be the grading on R with values in $F(\mathcal{C}_R/\tau)$ obtained from $\deg_{\mathcal{C}}$ via the natural mapping $\mathcal{C}_R \to \mathcal{C}_R/\tau$. By construction τ preserves this grading, as do ∇ and Δ, and hence by (6), D also preserves this grading.

4.3.4. Lemma. *The module A can be graded with values in $F(\mathcal{C}_R/\tau)$, such that for any $r \in R^+$ that is homogeneous with respect to the grading $\deg_{\mathcal{C}/\tau}$, all the irreducible constituents of $r.\alpha$ are homogeneous of degree $\deg_{\mathcal{C}/\tau} r$. For the induced grading on $R \otimes A$, the morphisms ∇_A and Δ_A preserve grading.*

Proof. From 4.3.2 it follows that the grading is well defined provided that for homogeneous $r, r' \in R^+$ with $\deg_{\mathcal{C}/\tau} r \neq \deg_{\mathcal{C}/\tau} r'$ the elements $r.\alpha$ and $r'.\alpha$ have no constituents in common. Now by (16) we have $\langle r.\alpha, r'.\alpha \rangle = \langle r, D(r')\alpha^*(\alpha)\rangle = \langle r, D(r')\rangle$, which is zero since $\deg_{\mathcal{C}/\tau} r \neq \deg_{\mathcal{C}/\tau} D(r')$, proving the condition. This grading on A will also be denoted by the symbol $\deg_{\mathcal{C}/\tau}$. That ∇_A preserves this grading follows from the definition and 4.3.1. Consequently its adjoint Δ_A also preserves the grading. □

Although we cannot further decompose the elementary twisted P.S.H.-module A over (R, τ), it is possible to view A as a P.S.-module over subalgebras of R. Let $S \subseteq R$ be a τ-stable P.S.H.-subalgebra of R (by the structure theory for P.S.H.-algebras such subalgebras correspond bijectively to τ-stable subsets of $\mathcal{C}_R$). Then we have an embedding $f: S \hookrightarrow R$, whose adjoint map f^* is the orthogonal projection $R \to S$, so we have $f^* \circ f = \mathrm{Id}_S$. We now apply 4.2.1 with $g = D_\tau$, and find that A becomes a twisted P.S.H.-module over (S, τ); we call this the restriction to S of A.

We shall call $a \in \Omega(A)$ cuspidal over S if it is cuspidal in the restriction to S of A. Since the projection f^* annihilates all homogeneous elements r of R for which some $\rho \in \mathcal{C}_R \setminus \mathcal{C}_S$ occurs in $\deg_{\mathcal{C}} r$, all $a \in \Omega(A)$ for which none of the τ-orbits of cuspidals in $\mathcal{C}_S$ occur in $\deg_{\mathcal{C}/\tau} a$ are cuspidal over S. Conversely, if some $\sigma \in \mathcal{C}_S$ occurs in $\deg_{\mathcal{C}/\tau} a$, then $\sigma^\star(a) \neq 0$, so that a is not cuspidal over S. Therefore the $\mathbf{Z}$-span of all cuspidals over S in A is itself a twisted P.S.H.-module over (S', τ), where S' is the P.S.H.-subalgebra of R with $\mathcal{C}_{S'} = \mathcal{C}_R \setminus \mathcal{C}_S$; in fact it is an elementary module with α as unique cuspidal. Denoting this S'-submodule of A as $A[S']$, we can describe the restriction of A to S as the direct sum over all $a \in \Omega(A[S'])$ of elementary modules with cuspidal a.

We now proceed to prove that A is isomorphic to the tensor product of the S-submodule $A[S]$ and the S'-submodule $A[S']$. The difficulty that prevents us from proving this right away is that there is no direct way to define a map $A[S] \otimes A[S'] \to A$: we cannot multiply elements of A among each other. If we do in fact have $A \cong A[S] \otimes A[S']$, then the irreducible element $\omega \otimes \omega' \in A[S] \otimes A[S']$ should be mapped to the unique common constituent of $s'.\omega$ and $s.\omega'$, where $s \in S$ and $s' \in S'$ are the products of cuspidals for which $\omega \leq s.\alpha$ and $\omega' \leq s'.\alpha$. However, we do not know yet whether such a common constituent is unique in general.

We study this situation more closely. Let $\omega \leq s.\alpha$ be as above. The elements α and $\omega \in \Omega(A[S])$ are both cuspidal over S' and therefore determine elementary S'-submodules of A, which are $A[S']$ and $A_\omega[S']$, say. We consider the restriction of $s^\star$ to $A_\omega[S']$.

4.3.5. Proposition. *The map $s^\star$ restricted to $A_\omega[S']$ has the following properties.*
(a) It is a map $A_\omega[S'] \to A[S']$.
(b) It is a positive graded morphism of S'-comodules.
(c) It is a morphism of S'-modules.
(d) Every $a \in \Omega(A[S'])$ is a constituent of $s^\star(b)$ for some $b \in A_\omega[S']^+$.
(e) For every $b \in \Omega(A_\omega[S'])$, the element $s^\star(b)$ has exactly one constituent.

Proof. Since ∇_A and Δ_A preserve the grading $\deg_{\mathcal{C}/\tau}$, part (a) follows from $\deg_{\mathcal{C}/\tau} \omega = \deg_{\mathcal{C}/\tau} s$. Part (b) follows immediately from remarks in 4.1. Now let $s' \in S'$ and $b \in A_\omega[S']$ be arbitrary, then by (11) we have $s^\star(s'.b) = \sum_{(s)} D(s_1)^*(s').s_2^\star(b) = s'.s^\star(b)$, since as a consequence of the grading, all terms $D(s_1)^*(s')$ with $s_1 \notin S_0 \cong \mathbf{Z}$ vanish; this proves (c). Any $a \in \Omega(A[S'])$ occurs as a constituent of $s'.\alpha$ for some $s' \in S'^+$, and since $s^\star(\omega) = \langle s.\alpha, \omega\rangle\alpha$ it occurs also as a constituent of $\langle s.\alpha, \omega\rangle s'.\alpha = s'.s^\star(\omega) = s^\star(s'.\omega)$, from which (d) follows. For two distinct elements $a, a' \in \Omega(A[S'])$ there cannot be a common constituent of $s.a$ and $s.a'$, since a and a' are both cuspidal over S, and 4.3.3 may be applied. This shows that for $b \in \Omega(A_\omega[S'])$ there can be at most one constituent of $s^\star(b)$; on the other hand $s^\star(b)$ cannot be zero since for some $s' \in S'$ we have $b \leq s'.\omega \leq s'.s.\alpha$. This proves part (e). □

We have shown for every $b \in \Omega(A_\omega[S'])$ that $s^\star(b)$ is a positive integer multiple of some $a \in \Omega(A[S'])$, and that every $a \in \Omega(A[S'])$ occurs at least once in this manner. We would like to prove that every a occurs exactly once, and that all the positive integer factors are equal, i.e., that $s^\star$ is an integer multiple of an isomorphism of twisted P.S.H.-modules over (S', τ). It appears however that to prove this we require a fact that will follow only from the classification of elementary twisted P.S.H.-modules over minimal P.S.H.-algebras given in the next section. At this point we consider it as a

4.3.6. Hypothesis. *For any $a \in \Omega(A)$ and $\rho \in \mathcal{C}_R$, all constituents of $\rho^\star(a)$ have multiplicity 1.*

To indicate the plausibility of this hypothesis, we show that the models encountered so far satisfy it.

4.3.7. Proposition. *The twisted P.S.H.-modules given in 4.2.2 both satisfy 4.3.6.*

Proof. Both examples are constructed by application of 4.2.1 to untwisted P.S.H.-modules. It is clear that in that context the operator $s^\star\colon A \to A$ adjoint to multiplication by $s \in S$ equals the operator $f(s)^\star$ adjoint to multiplication by $f(s) \in R$. So with $\rho \in R_E$ denoting the cuspidal element, we need to consider the operators $(\Delta_{R_E}\rho)^\star = (\rho \otimes 1)^\star + (1 \otimes \rho)^\star$ in the P.S.H.-module $R_E \otimes R_E$ over itself, and $(\nabla_{R_E}(\rho \otimes 1))^\star = (\nabla_{R_E}(1 \otimes \rho))^\star = \rho^\star$ in the P.S.H.-module R_E over itself. These operators coincide with $(\rho \otimes 1)^* + (1 \otimes \rho)^*$ and ρ^*, respectively, defined in the respective P.S.H.-algebras.

Now in [Zelev §4] it is shown that $\Omega(R_E)$ is naturally parametrised by the set $\mathcal{P}$ of partitions, where the subset $\mathcal{P}_n$ of partitions of $n \in \mathbf{N}$ corresponds to the set of irreducible elements of degree n. We denote the element parametrised by $\lambda \in \mathcal{P}$ as ω_λ (whereas Zelevinsky writes $\{\lambda\}$); we also write λ^- for the set of partitions obtainable from λ by removing one square from the Young diagram of λ. Then we obtain as a special case of Theorem 4.3 of [Zelev] that

$$\rho^*(\omega_\lambda) = \sum_{\mu \in \lambda^-} \omega_\mu. \tag{21}$$

Since the coefficients are all 1, this establishes the proposition for 4.2.2(I). Also, one deduces from (21) that

$$\bigl((\rho \otimes 1)^* + (1 \otimes \rho)^*\bigr)(\omega_\lambda \otimes \omega_\mu) = \sum_{\lambda' \in \lambda^-} \omega_{\lambda'} \otimes \omega_\mu + \sum_{\mu' \in \mu^-} \omega_\lambda \otimes \omega_{\mu'}, \tag{22}$$

establishing the proposition for 4.2.2(II). □

4.3.8. Lemma. *Under the assumption of 4.3.6, the morphism $s^\star$ of 4.3.5 is the k-fold multiple of an isomorphism of twisted P.S.H.-modules, where $k = \langle \omega, s.\alpha \rangle$.*

Proof. We show that there is a bijection $\Omega(A_\omega[S']) \to \Omega(A[S'])$ denoted by $a \mapsto \tilde{a}$ such that $s^\star(a) = k\tilde{a}$. We uniformly shift down the grading on $A_\omega[S']$ in such a way that $\deg \omega = 0$; then we must clearly have $\deg \tilde{a} = \deg a$, and we can construct the bijection by induction on this grading. In degree 0 the only irreducible elements of $A_\omega[S']$ and $A[S']$, respectively, are ω and α, so $\tilde{\omega} = \alpha$, and we have $s^\star(\omega) = k\alpha$ as required. Now let $a \in \Omega(A_\omega[S'])$, and assume that the bijection has been constructed in all degrees $< \deg a$. By 4.3.5.(e) there is a unique constituent of $s^\star(a)$, so define $\tilde{a}$ to be this constituent, and let k' be its multiplicity. Now for any $\rho \in \mathcal{C}_{S'}$ we have $s^\star(\rho^\star(a)) = \rho^\star(s^\star(a)) = k'\rho^\star(\tilde{a})$. By 4.3.6 we have $\rho^\star(a) = \sum_j a_j$ for some set of distinct $a_j \in \Omega(A_\omega[S'])$, and since these are of degree $\deg a - \deg \rho$ we may apply induction to obtain $s^\star(\rho^\star(a)) = k \sum_j \tilde{a}_j$. Since by 4.3.6 the constituents of $\rho^\star(\tilde{a})$ have multiplicity 1 it follows that $k' = k$ and $\rho^\star(\tilde{a}) = \sum_j \tilde{a}_j$.

By 4.3.5.(d) the map $a \mapsto \tilde{a}$ is surjective onto $\Omega(A[S'])$; its injectivity is proved as follows. First consider any product $p = r.b \in A_\omega[S']^+$ of homogeneous elements $r \in I$ and $b \in A$, and arbitrary $x \in A_\omega[S']$, where p and x are both of degree $\deg a$; then by 4.3.5.(b,c), $\langle s^\star(x), s^\star(p) \rangle = \langle s^\star(x), s^\star(r.b) \rangle = \langle s^\star(r^\star(x)), s^\star(b) \rangle$ which by induction is $k^2\langle r^\star(x), b \rangle = k^2\langle x, r.b \rangle = k^2\langle x, p \rangle$. In particular $\langle s^\star(p), s^\star(p) \rangle = k^2\langle p, p \rangle$. Now if we decompose p as $\sum_i c_i a_i$ with $c_i \in \mathbf{Z}$, $c_i > 0$ and distinct $a_i \in \Omega(A_\omega[S'])$, then $\langle p, p \rangle = \sum_i c_i^2$, and hence $\langle s^\star(p), s^\star(p) \rangle = k^2 \sum_i c_i^2$. On the other hand we have $s^\star(p) = \sum_i k c_i \tilde{a}_i$ whence $\langle s^\star(p), s^\star(p) \rangle = k^2 \sum_{i,j} c_i c_j \langle \tilde{a}_i, \tilde{a}_j \rangle$; since all these terms are positive it follows that $\tilde{a}_i \neq \tilde{a}_j$ for $i \neq j$. Taking $p = s'.\omega$ for an appropriate product s' of cuspidals, injectivity of $a \mapsto \tilde{a}$ follows. We have now proved that $s^\star$ is the k-fold multiple of an isometry; that this isometry is an isomorphism of twisted P.S.H.-modules follows from 4.3.5. □

4.3.9. Theorem. *Under the assumption of 4.3.6, the elementary twisted P.S.H.-module A is isomorphic to the tensor product $A[S] \otimes A[S']$.*

Proof. From 4.3.8 it follows that for $\omega' \leq s'.\alpha$ with $\omega' \in \Omega(A[S'])$ and s' a product of cuspidals in S', there is a unique common constituent of $s'.\omega$ and $s.\omega'$: it is the element mapped to $k\omega'$ by $s^\star$. We define a graded T-group morphism $f\colon A[S] \otimes A[S'] \to A$ by sending $\omega \otimes \omega'$ to this common constituent. By 4.3.8 f is an isometry (it is a bijection on the sets of irreducible elements), therefore the theorem will follow if f is a morphism of R-modules, as it will then also be a comodule morphism by adjunction. Since any element of $R \cong S \otimes S'$ is a sum of products of elements of S and S', it will be sufficient to prove that f commutes with multiplication by such elements. For elements of S' this follows from 4.3.5.(c); for elements of S we may interchange the rôles of S and S', since this leaves the definition of f invariant. □

4.3.10. Corollary. *Assuming 4.3.6, the twisted P.S.H.-module A is isomorphic to the tensor product $\bigotimes_{\bar\rho\in\mathcal{C}_R/\tau} A[R_{\bar\rho}]$, where $R_{\bar\rho}$ is the (minimal τ-stable) P.S.H.-subalgebra of R whose cuspidals form the τ-orbit $\bar\rho$.*

Proof. In the case where $\mathcal{C}_R/\tau$ is finite, this follows immediately by induction from 4.3.9. This also shows that in the infinite case each finite tensor product of modules $A[R_{\bar\rho}]$ can be embedded in A as a submodule $A[R']$ with $\mathcal{C}_{R'}/\tau$ finite; moreover one easily checks that these embeddings are compatible with the embeddings of finite tensor products into larger ones (extending tensors with tensorands α). Now the corollary follows since A is the direct limit of such submodules $A[R']$, and the infinite tensor product is similarly the direct limit of the finite ones. □

§5. Classification of elementary twisted P.S.H.-modules.

We classify the elementary twisted P.S.H.-modules over all instances of minimal τ-stable P.S.H.-algebras, of which there are three kinds, one of which can be easily seen to allow no non-zero twisted P.S.H.-modules. For the other two cases the classification takes the form of a uniqueness result (similarly to the case for elementary P.S.H.-algebras), where the standard models are those of 4.2.2.

5.1. *Uniqueness of elementary twisted P.S.H.-modules over* $(R_E\otimes R_E,\chi)$.

Let A be an elementary twisted P.S.H.-module over $(R_E\otimes R_E,\chi)$ with cuspidal element α; we shall show that it is isomorphic to the model in 4.2.2(I). As a conseqence of 4.1.3 this is rather easy. We first show that any multiplication on A by an element of $R_E\otimes R_E$ is equivalent to multiplication by an element of the form $r\otimes 1$.

5.1.1. Proposition. *Let $p\colon R_E\otimes R_E\to R_E\otimes R_E$ be given by $r\otimes s\mapsto rs\otimes 1$, and let p^* be its adjoint. Then $\nabla_A=\nabla_A\circ(p\otimes \mathrm{Id}_A)$ and $\Delta_A=(p^*\otimes\mathrm{Id}_A)\circ\Delta_A$.*

Proof. For $r,s\in R_E$ and $a\in A$ we have $(r\otimes s).a=(r\otimes 1).(1\otimes s).a=(r\otimes 1).(s\otimes 1).a=(rs\otimes 1).a$ by 4.1.3, proving the first statement. The second statement follows by adjunction. □

Let $f\colon R_E\to R_E\otimes R_E$ be the embedding $x\mapsto x\otimes 1$; this is a P.S.H.-algebra morphism, as is ∇_{R_E}, and therefore $p=f\circ\nabla_{R_E}$ is a P.S.H.-algebra endomorphism of $R_E\otimes R_E$. The proposition therefore states that applying the construction of 4.2.1 to A for the endomorphism p leads to an identical P.S.-module, i.e., $\nabla'_A=\nabla_A$ and $\Delta'_A=\Delta_A$. We factor this construction into two steps according to $p=f\circ\nabla_{R_E}$. Let $f^*: R_E\otimes R_E\to R_E$ be the adjoint $\mathrm{Id}_{R_E}\otimes e^*$ of f. Applying 4.2.1 for f and $g=D_\chi$ we make A into a P.S.-module over R_E; since we have seen in 4.2 that $D_\chi=\Delta_{R_E}\circ\nabla_{R_E}$, we have

$$f^*\circ D_\chi\circ f=f^*\circ\Delta_{R_E}\circ\nabla_{R_E}\circ f=\mathrm{Id}_{R_E}\circ\mathrm{Id}_{R_E}=\mathrm{Id}_{R_E}\tag{23}$$

by the unit and counit properties of R_E, so A is in fact an (untwisted) P.S.H.-module over R_E. Apparently the twisted P.S.H.-module A over $R_E\otimes R_E$ can be obtained from this P.S.H.-module over R_E by applying 4.2.1 for ∇_{R_E} and $g=\mathrm{Id}_{R_E}$.

5.1.2. Theorem. *Every elementary twisted P.S.H.-module over $(R_E\otimes R_E,\chi)$ is isomorphic to the one in 4.2.2(I).*

Proof. Since the twisted P.S.H.-module structures of A and of the module R_E of 4.2.2(I) are obtained by the same construction from P.S.H.-module structures over R_E, it is sufficient to prove these underlying P.S.H.-modules to be isomorphic. Cuspidal elements remain cuspidal under application of the construction of 4.2.1, so A is an elementary P.S.H.-module with unique cuspidal α; define a map $\phi\colon R_E\to A$ by $\phi(x)=x.\alpha$. Clearly ϕ is a morphism of T-groups and of R_E-modules. For P.S.H.-modules one has the formulae of 4.1 with D replaced by identity, so we have

$$\langle\phi(r),\phi(s)\rangle=\langle r.\alpha,s.\alpha\rangle=\langle\alpha^*(r.\alpha),s\rangle=\langle r.\alpha^*(\alpha),s\rangle=\langle r,s\rangle,\tag{24}$$

and ϕ is an isometry. Therefore the image of ϕ is a P.S.H.-submodule of A, and since A is elementary this means ϕ is surjective. The adjoint map ϕ^* of ϕ therefore equals ϕ^{-1}, and since this is a morphism of R_E-comodules, so is ϕ, and ϕ is an isomorphism of P.S.H.-modules. □

Since 4.3.6 holds for the model of 4.2.2(I) by 4.3.7, we have now proved that it holds for elementary twisted P.S.H.-modules over $(R_E\otimes R_E,\chi)$, and by 4.3.3 for arbitrary twisted P.S.H.-modules over $(R_E\otimes R_E,\chi)$. In fact 4.3.6 is now proved for all cases where ρ is not stable under τ, since the operator $\rho^\star$ is not affected by restriction of A to $R(\rho)\otimes R(\tau(\rho))$ (because multiplication by ρ means the same thing), and by the structure theory of P.S.H.-algebras this P.S.H-subalgebra is isomorphic to $R_E\otimes R_E$, with τ acting as χ.

5.2. *Uniqueness of elementary twisted P.S.H.-modules over* (R_E, Id)**.**

Now let A be an elementary twisted P.S.H.-module over (R_E, τ) for some τ with cuspidal element α; we shall soon see that τ can only be identity. Our goal is to prove the analogue of 5.1.2, but this will be considerably more difficult. We want to show that A can be derived from an untwisted P.S.H.-module over $R_E \otimes R_E$, but in this case the untwisted structure cannot be retrieved from the twisted P.S.H.-module structure; indeed the latter structure admits an additional symmetry that would require interchange of the two tensorands R_E in the former structure. Therefore, we cannot avoid doing some rather technical calculations. However, the example in 4.2.2(II) may serve as a model for the structure we expect to find and so clarify some of the assertions we make. In the sequel we write $R = R_E$.

Since $\tau(\rho) = \rho$ it follows from (15) that for $a, b \in A$ we have the useful formula

$$\langle \rho.a, \rho.b \rangle = 2\langle a, b \rangle + \langle \rho^\star(a), \rho^\star(b) \rangle. \tag{25}$$

A first consequence is $\langle \rho.\alpha, \rho.\alpha \rangle = 2$, so that $\rho.\alpha$ must be the sum of two distinct constituents, say ζ and η. By 4.3.2, $\Omega(A_0) = \{\alpha\}$ and $\Omega(A_1) = \{\zeta, \eta\}$, and we have $\Delta_A \zeta = 1 \otimes \zeta + \rho \otimes \alpha$ and $\Delta_A \eta = 1 \otimes \eta + \rho \otimes \alpha$, so that $\rho^\star(\zeta) = \rho^\star(\eta) = \alpha$ and $\alpha^*(\zeta) = \alpha^*(\eta) = \rho$. There is no feature that distinguishes η from ζ, so we may always apply the symmetry of interchanging ζ and η to any facts derived. Applying (25) to ζ and η we find $\langle \rho.\zeta, \rho.\zeta \rangle = \langle \rho.\eta, \rho.\eta \rangle = 3$ and $\langle \rho.\zeta, \rho.\eta \rangle = 1$. The first fact means that both $\rho.\zeta$ and $\rho.\eta$ have three distinct constituents, the latter fact that they have exactly one constituent in common; we call this constituent ξ. Since $\rho.\alpha = \zeta + \eta$, it follows from 4.3.2 that $\Omega(A_2)$ consists of the five irreducible constituents thus found.

Recall from the proof of 4.3.7 that $\Omega(R_n)$ consists of the elements ω_λ for $\lambda \in \mathcal{P}_n$. So we can find further elements of A_2 by multiplying α by the elements $\omega_{(2)}, \omega_{(1,1)} \in \Omega(R_2)$. In the following computations we use the facts in [Zelev 3.1], where $\omega_{(2)}$ and $\omega_{(1,1)}$ are called x_2 and y_2 respectively. We have $\langle \omega_{(2)}.\alpha, \omega_{(2)}.\alpha \rangle = \langle \alpha^*(\omega_{(2)}.\alpha), \omega_{(2)} \rangle = \langle D(\omega_{(2)}), \omega_{(2)} \rangle$ by (16), while $\Delta\omega_{(2)} = \omega_{(2)} \otimes 1 + \rho \otimes \rho + 1 \otimes \omega_{(2)}$. There are two P.S.H.-algebra automorphisms of R, of which the non-trivial one exchanges $\omega_{(2)}$ and $\omega_{(1,1)}$. Suppose τ is this non-trivial automorphism, then we have $D(\omega_{(2)}) = \omega_{(2)} + \rho^2 + \omega_{(1,1)} = 2\rho^2$ so that $\langle D(\omega_{(2)}), \omega_{(2)} \rangle = 2$ and $\omega_{(2)}.\alpha$ has 2 constituents. Also $\omega_{(2)}.\alpha = \omega_{(1,1)}.\alpha$ by 4.1.3, and all elements of $\Omega(A_2)$ occur among its constituents by 4.3.2, in plain contradiction with $\#\Omega(A_2) = 5$. Therefore τ must be identity, so that $D(\omega_{(2)}) = 2\omega_{(2)} + \rho^2$ and $\langle D(\omega_{(2)}), \omega_{(2)} \rangle = 3$. We find that $\omega_{(2)}.\alpha$ has three distinct constituents, and this holds similarly for $\omega_{(1,1)}.\alpha$. We also have $\langle \omega_{(2)}.\alpha, \rho.\zeta \rangle = \langle \omega_{(2)}, \alpha^*(\rho.\zeta) \rangle = \langle \omega_{(2)}, D(\rho).\rho \rangle = \langle \omega_{(2)}, 2\rho^2 \rangle = 2$, and similarly $\langle \omega_{(2)}.\alpha, \rho.\eta \rangle = 2$, meaning that $\omega_{(2)}.\alpha$ has a pair of constituents in common with both $\rho.\zeta$ and $\rho.\eta$. These two pairs must have a non-empty intersection, which can only be $\{\xi\}$; we call the remaining elements of these pairs $\zeta_{(2)}$ and $\eta_{(2)}$ respectively. In a similar way ξ is a constituent of $\omega_{(1,1)}.\alpha$ as are the two yet unnamed elements of $\Omega(A_2)$, which we denote $\zeta_{(1,1)}$ and $\eta_{(1,1)}$ according as they are a constituent of $\rho.\zeta$ or $\rho.\eta$.

Now let A_ζ be the T-subgroup of A generated by $\{\, a \in \Omega(A) \mid \eta^*(a) = 0 \,\}$. Of the irreducible elements found so far, we have $\alpha, \zeta, \zeta_{(2)}, \zeta_{(1,1)} \in A_\zeta$. Since $\eta \in A^+$ we have for any $a \in A^+$ with $\eta^*(a) = 0$ that $a \in A_\zeta$. If moreover $r \in R^+$ then $\eta^*(r^\star(a)) = r^*(\eta^*(a)) = 0$ while also $r^\star(a) \in A^+$, so that $r^\star(a) \in A_\zeta$. Since R^+ generates R as additive group, we find that A_ζ is closed under all operators $r^\star$ and is therefore a sub-comodule of A over R; the same is true for A_η defined similarly.

5.2.1. Lemma. *The restriction* $\alpha^*\colon A_\zeta \to R$ *is an isomorphism of T-groups and of comodules over* R*, and the same holds for* $\alpha^*\colon A_\eta \to R$*.*

Proof. We give the proof for A_ζ. As remarked in 4.1 the operator a^* is a comodule morphism $A \to R$, so any restriction to a sub-comodule is a morphism of comodules. It remains to prove that the restriction is a T-group isomorphism, i.e., that it defines a bijection on the sets of irreducible elements. We prove this by induction on the degree; since $\Omega(R_n)$ is parametrised by $\mathcal{P}_n$, this amounts to proving the following assertion by induction on n.

$P(n)$: For each $\lambda \in \mathcal{P}_n$ there exists an element $\zeta_\lambda \in \Omega(A_\zeta)$ such that $\alpha^*(\zeta_\lambda) = \omega_\lambda$, and ζ_λ is the unique constituent in A_ζ of $\omega_\lambda.\alpha$.

Putting $\zeta_{(0)} = \alpha$ and $\zeta_{(1)} = \zeta$, the statements $P(0)$, $P(1)$ and $P(2)$ follow from the explicit computations above. So assume that $n > 2$ and that $P(m)$ holds for $m < n$; let $\lambda \in \mathcal{P}_n$.

First we show, for any $\mu \in \lambda^-$, that all constituents in A_ζ of $\omega_\lambda.\alpha$ occur among those of $\rho.\zeta_\mu$. Let $a \in \Omega(A_\zeta)$ be a constituent of $\omega_\lambda.\alpha$, then ω_λ occurs as a constituent of $\alpha^*(a)$, and therefore all constituents of $\rho^*(\omega_\lambda)$—which include ω_μ by (21)—occur among those of $\rho^*(\alpha^*(a)) = \alpha^*(\rho^\star(a))$. By $P(n-1)$ this means that $\rho^\star(a)$ has a constituent ζ_μ, so a is a constituent of $\rho.\zeta_\mu$ as claimed.

Now for any $\mu \in \mathcal{P}_{n-1}$ we have by (13) that $\eta^*(\rho.\zeta_\mu) = D(\rho).\eta^*(\zeta_\mu) + D(\rho^*(\rho)).\alpha^*(\zeta_\mu) = \omega_\mu$ which is irreducible, so of the irreducible constituents of $\rho.\zeta_\mu$ all but one vanish under η^*, and the remaining one, say ξ_μ, has multiplicity 1 and satisfies $\eta^*(\xi_\mu) = \omega_\mu$. For the projection $p_\mu = \rho.\zeta_\mu - \xi_\mu$ of $\rho.\zeta_\mu$ on A_ζ we have $p_\mu \perp \xi_\mu$, and so $\langle \rho.\zeta_\mu, \rho.\zeta_\mu \rangle = \langle p_\mu, p_\mu \rangle + 1$. Of this we can compute the left hand side using (25) and the fact that α^* is an isometry in degree $n-2$: $\langle \rho.\zeta_\mu, \rho.\zeta_\mu \rangle = 2\langle \zeta_\mu.\zeta_\mu \rangle + \langle \rho^\star(\zeta_\mu), \rho^\star(\zeta_\mu) \rangle = 2 + \langle \rho^*(\omega_\mu), \rho^*(\omega_\mu) \rangle = 2 + \#\mu^-$. The cardinality of μ^- is one less than that of $\mu^+ = \{\, \lambda \mid \mu \in \lambda^- \,\}$, since for every row of the Young diagram of μ that we can take away a square from, we might also *add* a square to the next row of μ, and we can always add a square to its first row. Therefore we obtain

$$\langle p_\mu, p_\mu \rangle = \#\mu^+. \tag{26}$$

Note that (21) may also be written as $\rho.\omega_\mu = \sum_{\lambda \in \mu^+} \omega_\lambda$. We shall determine below for every constituent of $\rho.\omega_\mu$ a corresponding one of p_μ, with multiplicity 1; then by (26) we shall have found all such constituents.

Let $\lambda \in \mathcal{P}_n$, and suppose first that $\#\lambda^- \geq 2$. For such λ choose two distinct partitions $\mu, \mu' \in \lambda^-$; one easily checks that $\mu^+ \cap \mu'^+ = \{\lambda\}$ and $\#(\mu^- \cap \mu'^-) = 1$. We first show that p_μ and $p_{\mu'}$ have a unique common constituent; since we computed above that $\eta^*(\xi_\mu) = \omega_\mu$, which implies $\xi_\mu \neq \xi_{\mu'}$, it suffices to show that $\rho.\zeta_\mu$ and $\rho.\zeta_{\mu'}$ have a unique common constituent. We have $\langle \rho.\zeta_\mu, \rho.\zeta_{\mu'} \rangle = 2\langle \zeta_\mu, \zeta_{\mu'} \rangle + \langle \rho^\star(\zeta_\mu), \rho^\star(\zeta_{\mu'}) \rangle = \langle \rho^*(\omega_\mu), \rho^*(\omega_{\mu'}) \rangle = \#(\mu^- \cap \mu'^-) = 1$, so there is indeed a unique common constituent of p_μ and $p_{\mu'}$, and ζ_λ is defined to be this common constituent. It remains to verify that $\alpha^*(\zeta_\lambda) = \omega_\lambda$: it follows from the remarks above that $\omega_\lambda.\alpha$ can have no constituents in A_ζ other than ζ_λ, and the independence of ζ_λ of the choice of the pair μ, μ' will also follow. We have the inequality $\alpha^*(\zeta_\lambda) \leq \alpha^*(\rho.\zeta_\mu) = D(\rho).\alpha^*(\zeta_\mu) = 2\rho.\omega_\mu = 2\sum_{\nu \in \mu^+} \omega_\nu$, and a similar one for μ'; these two can be combined to $\alpha^*(\zeta_\lambda) \leq 2\omega_\lambda$. On the other hand we have, using that α^* is isometric in degree $n-1$, that $\langle \alpha^*(\zeta_\lambda), \rho.\omega_\mu \rangle = \langle \rho^*(\alpha^*(\zeta_\lambda)), \omega_\mu \rangle = \langle \alpha^*(\rho^\star(\zeta_\lambda)), \omega_\mu \rangle = \langle \rho^\star(\zeta_\lambda), \zeta_\mu \rangle = \langle \zeta_\lambda, \rho.\zeta_\mu \rangle = 1$, which leaves as the only remaining possibility that $\alpha^*(\zeta_\lambda) = \omega_\lambda$, completing the proof of $P(n)$ for this λ.

Now suppose $\#\lambda^- = 1$, say $\lambda^- = \{\mu\}$; this means that λ is a 'rectangular' partition, i.e., all non-zero parts of λ are equal. Since $n > 2$ one easily verifies that there can be no other such partitions in μ^+, so we may assume that appropriate elements $\zeta_{\lambda'}$ have been defined for all $\lambda' \in \mu^+ \setminus \{\lambda\}$. These elements are by construction constituents of p_μ with multiplicity 1, and because $\alpha^*(\zeta_{\lambda'}) = \omega_{\lambda'}$ they are distinct, so by (26) there is exactly one remaining constituent of p_μ, and it has multiplicity 1; we define ζ_λ to be this constituent. We have shown for $\lambda' \in \mu^+ \setminus \{\lambda\}$ that $\omega_{\lambda'}.\alpha$ has no constituents in A_ζ other than $\zeta_{\lambda'}$, so in particular $\langle \alpha^*(\zeta_\lambda), \omega_{\lambda'} \rangle = 0$. On the other hand the fact that ζ_λ is a constituent of p_μ with multiplicity 1 implies by the same computations as those above that $\alpha^*(\zeta_\lambda) \leq 2\sum_{\nu \in \mu^+} \omega_\nu$ and $\langle \alpha^*(\zeta_\lambda), \rho.\omega_\mu \rangle = 1$, so that $\alpha^*(\zeta_\lambda)$ must equal ω_λ. Similarly, as we have shown that any constituents of $\omega_\lambda.\alpha$ in A_ζ must occur among the elements $\zeta_{\lambda'}$ for $\lambda' \in \mu^+$, but also that for such λ' we have $\alpha^*(\zeta_{\lambda'}) = \omega_{\lambda'}$ it follows that ζ_λ is the only such constituent. This completes the proof of $P(n)$, and of the lemma. □

We define elements $\eta_\lambda \in \Omega(A_\eta)$ similarly to ζ_λ. Since $\zeta + \eta = \rho.\alpha$ and by 4.3.2, α is the only element of $\Omega(A)$ that vanishes under $(\rho.\alpha)^*$, we have $A_\zeta \cap A_\eta = \mathbf{Z}\alpha$. For $r \in R$ and $a \in A_\zeta$ we have $r^\star(a) \in A_\zeta$, and hence we have for any non-zero partition $\lambda \in \mathcal{P}$ that $0 = \langle r^\star(a), \eta_\lambda \rangle = \langle \eta^*_\lambda(a), r \rangle$, whence we must have $\eta^*_\lambda(a) = 0$; so not only η^* but any η^*_λ with $\lambda \neq (0)$ vanishes on A_ζ. Since $\Delta_A \eta_\lambda \in R \otimes A_\eta$ this means that in the following application of (13) the sum reduces to a single term:

$$\eta^*_\lambda(r.\zeta_\mu) = D(\omega^*_\lambda(r)).\alpha^*(\zeta_\mu) = D(\omega^*_\lambda(r)).\omega_\mu \tag{27}$$

for all $\lambda, \mu \in \mathcal{P}$ and $r \in R$. We similarly have

$$\zeta^*_\lambda(r.\eta_\mu) = D(\omega^*_\lambda(r)).\omega_\mu. \tag{28}$$

Remark. In the model of 4.2.2(II) the elements ζ_λ and η_λ are respectively $\omega_\lambda \otimes 1$ and $1 \otimes \omega_\lambda$ (assuming the choice $\zeta = \rho \otimes 1$, $\eta = 1 \otimes \rho$); we may informally view A_ζ and A_η as opposite 'sides' of A.

We shall now determine all elements of $\Omega(A)$. Considering 4.2.2(II) it should come as no surprise that $\Omega(A_n)$ will be parametrised by $\bigcup_{i+j=n} \mathcal{P}_i \times \mathcal{P}_j$; we shall construct for each pair of partitions (λ, μ) a distinct irreducible element denoted $\alpha_{\lambda,\mu}$. To prove that the $\alpha_{\lambda,\mu}$ exhaust $\Omega(A_n)$, we apply the following line of reasoning. We know that all elements of $\Omega(A_n)$ are constituents of $\rho^n.\alpha$, and that the sum of the squares of their multiplicities is $\langle \rho^n.\alpha, \rho^n.\alpha \rangle$, which we can compute to be $\langle \alpha^*(\rho^n.\alpha), \rho^n \rangle = \langle D(\rho^n).\rho^n \rangle =$

$\langle \Delta\rho^n, \Delta\rho^n \rangle$, which is the sum of squares of the multiplicities of the constituents $\omega_\lambda \otimes \omega_\mu$ of $\Delta\rho^n$ (for $(\lambda, \mu) \in \bigcup_{i+j=n} \mathcal{P}_i \times \mathcal{P}_j$). So if we can prove the multiplicity $\langle \alpha_{\lambda,\mu}, \rho^n.\alpha \rangle$ to be at least equal to the multiplicity $m(\lambda, \mu)$ of $\omega_\lambda \otimes \omega_\mu$ in $\Delta\rho^n$, then it will follow that the $\alpha_{\lambda,\mu}$ exhaust $\Omega(A)$, and that the multiplicities mentioned are in fact equal. We have a recursion formula for $m(\lambda, \mu)$:

$$\begin{aligned} m(\lambda, \mu) &= \langle \omega_\lambda \otimes \omega_\mu, \Delta\rho^n \rangle = \langle \omega_\lambda.\omega_\mu, \rho^n \rangle = \langle \rho^*(\omega_\lambda.\omega_\mu), \rho^{n-1} \rangle = \langle \rho^*(\omega_\lambda).\omega_\mu, \rho^{n-1} \rangle + \langle \omega_\lambda.\rho^*(\omega_\mu), \rho^{n-1} \rangle \\ &= \sum_{\lambda' \in \lambda^-} \langle \omega_{\lambda'}.\omega_\mu, \rho^{n-1} \rangle + \sum_{\mu' \in \mu^-} \langle \omega_\lambda.\omega_{\mu'}, \rho^{n-1} \rangle = \sum_{\lambda' \in \lambda^-} m(\lambda', \mu) + \sum_{\mu' \in \mu^-} m(\lambda, \mu'). \end{aligned}$$

Therefore, if we can prove the inequality

$$\rho^\star(\alpha_{\lambda,\mu}) \geq \sum_{\lambda' \in \lambda^-} \alpha_{\lambda',\mu} + \sum_{\mu' \in \mu^-} \alpha_{\lambda,\mu'}, \tag{29}$$

then the desired inequality $\langle \alpha_{\lambda,\mu}, \rho^n.\alpha \rangle \geq m(\lambda, \mu)$ will follow by an immediate induction on n.

We proceed to define the elements $\alpha_{\lambda,\mu}$. Applying (28) for $r = \omega_\lambda$ we find $\zeta_\lambda^*(\omega_\lambda.\eta_\mu) = \omega_\mu$. It follows that the positive elements $\omega_\lambda.\eta_\mu$ and $\omega_\mu.\zeta_\lambda$ have a unique constituent in common, and we define $\alpha_{\lambda,\mu}$ to be this common constituent. It follows from (27) and (28) that

$$\eta_\mu^*(\alpha_{\lambda,\mu}) = \omega_\lambda \quad \text{and} \quad \zeta_\lambda^*(\alpha_{\lambda,\mu}) = \omega_\mu. \tag{30a, b}$$

We define for partitions λ, μ the relation $\mu \subseteq \lambda$ to mean the corresponding inclusion of their Young diagrams. Then by [Zelev 4.13] we have $\omega_\mu^*(\omega_\lambda) = 0$ unless $\mu \subseteq \lambda$ (this also follows easily from (21)). Therefore, by another application of (27) and (28) we have

$$\eta_\nu^*(\alpha_{\lambda,\mu}) = 0 \quad \text{unless } \nu \subseteq \mu \qquad \text{and} \qquad \zeta_\nu^*(\alpha_{\lambda,\mu}) = 0 \quad \text{unless } \nu \subseteq \lambda. \tag{31a, b}$$

These relations imply that $\alpha_{\lambda,\mu} \neq \alpha_{\lambda',\mu'}$ for $(\lambda, \mu) \neq (\lambda', \mu')$.

5.2.2. Theorem. $\Omega(A) = \{ \alpha_{\lambda,\mu} \mid \lambda, \mu \in \mathcal{P} \}$, *and* $\deg \alpha_{\lambda,\mu} = |\lambda| + |\mu|$. *Moreover*

$$\rho^\star(\alpha_{\lambda,\mu}) = \sum_{\lambda' \in \lambda^-} \alpha_{\lambda',\mu} + \sum_{\mu' \in \mu^-} \alpha_{\lambda,\mu'}. \tag{32}$$

Proof. The second assertion is evident, and as indicated above it suffices to prove the inequality (29) for all λ, μ to obtain the first assertion, and then the inequality will also become an equation: (32). For any $\lambda' \in \lambda^-$ we have $\rho^\star(\omega_\mu.\zeta_\lambda) \geq \omega_\mu.\rho^\star(\zeta_\lambda) \geq \omega_\mu.\zeta_{\lambda'} \geq \alpha_{\lambda',\mu}$. On the other hand $\alpha_{\lambda',\mu}$ is not a constituent of the positive element $\rho^\star(\omega_\mu.\zeta_\lambda - \alpha_{\lambda,\mu})$ since $\eta_\mu^*(\alpha_{\lambda',\mu}) = \omega_{\lambda'} \neq 0$ while $\eta_\mu^*(\rho^\star(\omega_\mu.\zeta_\lambda - \alpha_{\lambda,\mu})) = \rho^*(\eta_\mu^*(\omega_\mu.\zeta_\lambda - \alpha_{\lambda,\mu})) = \rho^*(\omega_\lambda - \omega_\lambda) = 0$. Therefore $\alpha_{\lambda',\mu}$ is a constituent of $\rho^\star(\alpha_{\lambda,\mu})$, and we similarly prove that it has constituents $\alpha_{\lambda,\mu'}$ for all $\mu' \in \mu^-$; together this yields (29). □

We have proved that the linear map $\phi\colon A \to R \otimes R$ defined by $\alpha_{\lambda,\mu} \mapsto \omega_\lambda \otimes \omega_\mu$ is an isomorphism of graded T-groups, which by (22) and (32) commutes with the operator $\rho^\star$, and hence also with multiplication by ρ. Note in particular that A satisfies 4.3.6; by arguments similar to those given above this generalises to any instance of 4.3.6 where the cuspidal element ρ is fixed by τ, and therefore we have now proved that 4.3.6 holds for arbitrary twisted P.S.H.-modules. In order to prove that ϕ commutes with multiplication by arbitrary elements of R, i.e., that it is an isomorphism of modules (and of comodules), we have to use more detailed information about R.

5.3. *Uniqueness of elementary twisted P.S.H.-modules over* (R_E, Id) *(continuation).*

We borrow some notation from [Zelev]. For a partition $\lambda = (l_1, \ldots, l_r)$ we denote by $\lambda^\uparrow$ the partition $(l_2, \ldots, l_r)$ obtained by removing its first part; in the Young diagram this means removing the first row and then shifting the remaining rows one place up. For $\mu \in \mathcal{P}$ we define $\mu \perp \lambda$ to mean $\lambda^\uparrow \subseteq \mu \subseteq \lambda$ (there should be no confusion with 'perpendicular'); this means that the Young diagram of μ can be obtained from that of λ by removing at most one square from each column. As in [Zelev] the notation x_n will be used for the element $\omega_{(n)} \in \Omega(R_n)$. According to [Zelev 3.1(d)] it satisfies

$$\Delta x_n = \sum_{i+j=n} x_i \otimes x_j, \tag{33}$$

which implies in particular that the T-group R_x generated by all x_n is a sub-comodule of R over R. We shall also use the important formula in [Zelev 4.3],

$$\sum_{n\geq 0} x_n^*(\omega_\lambda) = \sum_{\mu\perp\lambda} \omega_\mu, \tag{34}$$

of which (15) is only a special case (the component for $n = 1$). By 5.2.1 this equation may be translated for A_ζ to

$$\sum_{n\geq 0} x_n^\star(\zeta_\lambda) = \sum_{\mu\perp\lambda} \zeta_\mu. \tag{35}$$

We start with deriving from this a less complete statement about A.

5.3.1. Lemma. *Any constituent $\alpha_{\lambda',\mu'}$ of $\sum_n x_n^\star(\alpha_{\lambda,\mu})$ satisfies $\lambda' \perp \lambda$ and $\mu' \perp \mu$.*

Proof. Repeated application of (32) shows that $\lambda' \subseteq \lambda$ and $\mu' \subseteq \mu$. We also have, using (33) and (35), that $\sum_n x_n^\star(\alpha_{\lambda,\mu}) \leq \sum_n x_n^\star(\omega_\mu.\zeta_\lambda) = \sum_{i,j} D(x_i)^*(\omega_\mu).x_j^\star(\zeta_\lambda) = \sum_{\nu\perp\lambda}(\sum_i D(x_i)^*(\omega_\mu)).\zeta_\nu$. Therefore $\alpha_{\lambda',\mu'}$ is a constituent of a summand of this sum for some $\nu \perp \lambda$, and since that summand contains a factor ζ_ν we conclude that $\zeta_\nu^*(\alpha_{\lambda',\mu'}) \neq 0$, which by (31b) implies $\lambda^\uparrow \subseteq \nu \subseteq \lambda' \subseteq \lambda$, whence $\lambda' \perp \lambda$. The proof of $\mu' \perp \mu$ is similar (using $\omega_\lambda.\eta_\mu$ instead of $\omega_\mu.\zeta_\lambda$). □

5.3.2. Lemma. *If l is the first part of λ, then $\alpha_{\lambda,\mu}$ is a constituent of $x_l.\alpha_{\lambda^\uparrow,\mu}$ with multiplicity 1.*

Proof. We prove in a way similar to the proof of 5.2.2 that $\alpha_{\lambda^\uparrow,\mu}$ is a constituent of $x_l^\star(\alpha_{\lambda,\mu})$. On one hand $\alpha_{\lambda^\uparrow,\mu}$ is a constituent of $x_l^\star(\omega_\mu.\zeta_\lambda)$ since $x_l^\star(\omega_\mu.\zeta_\lambda) \geq \omega_\mu.x_l^\star(\zeta_\lambda) = \omega_\mu.\zeta_{\lambda^\uparrow}$, but it is not a constituent of $x_l^\star(\omega_\mu.\zeta_\lambda - \alpha_{\lambda,\mu})$ since $\eta_\mu^*(x_l^\star(\omega_\mu.\zeta_\lambda - \alpha_{\lambda,\mu})) = x_l^*(\eta_\mu^*(\omega_\mu.\zeta_\lambda - \alpha_{\lambda,\mu})) = x_l^*(\omega_\lambda - \omega_\lambda) = 0$ while $\eta_\mu^*(\alpha_{\lambda^\uparrow,\mu}) = \omega_{\lambda^\uparrow} \neq 0$. From the latter computation we also conclude $\eta_\mu^*(x_l^\star(\alpha_{\lambda,\mu})) = x_l^*(\omega_\lambda) = \omega_{\lambda^\uparrow}$, whence the multiplicity of $\alpha_{\lambda^\uparrow,\mu}$ in $x_l^\star(\alpha_{\lambda,\mu})$ does not exceed 1. The lemma follows. □

In the proof that ϕ is a morphism of (co-)modules, we employ a particular partial ordering on $\mathcal{P}$, denoted by '$\leq_1$'. By definition $\lambda <_1 \mu$ holds if either $|\lambda| < |\mu|$ or $|\lambda| = |\mu|$ and $l > m$ for the first parts l and m of λ and μ respectively (note the order reversal); as usual $\lambda \leq_1 \mu$ means that $\lambda <_1 \mu$ or $\lambda = \mu$. The relevant property of this ordering is the following. Note first that $\lambda^\uparrow \perp \mu$ is a strictly weaker condition than $\mu \perp \lambda$ because the former condition puts no upper limit on the first part of μ. However, we do have

$$\lambda^\uparrow \perp \mu \wedge |\mu| \leq |\lambda| \;\Rightarrow\; \mu \leq_1 \lambda. \tag{36}$$

For, if $\lambda^\uparrow \perp \mu$ and $|\lambda| = |\mu|$ then either $m > l$ or $\mu = \lambda$. We obtain the following complement to the previous lemma.

5.3.3. Lemma. *Any constituent $\alpha_{\lambda',\mu'}$ of $x_l.\alpha_{\lambda^\uparrow,\mu}$ satisfies either $\lambda' <_1 \lambda$ or $(\lambda',\mu') = (\lambda,\mu)$.*

Proof. From 5.3.1 and (36) we get $\lambda' \leq_1 \lambda$, while if $\lambda' = \lambda$ we have $|\mu'| = |\mu|$, implying $\mu' = \mu$. □

We are now ready to prove that ϕ is an isomorphism of twisted P.S.H.-modules. In the proof we shall often implicitly use the following reasoning. All facts derived for A automatically apply to the twisted P.S.H.-module $R \otimes R$ of 4.2.2(II), with $\omega_\lambda \otimes \omega_\mu$ playing the rôle of $\alpha_{\lambda,\mu}$ in $R \otimes R$. Therefore, if we have proved a statement in A then we can obtain a true statement in $R \otimes R$ by replacing every $\alpha_{\lambda,\mu}$ by $\omega_\lambda \otimes \omega_\mu = \phi(\alpha_{\lambda,\mu})$.

5.3.4. Theorem. *The map $\phi\colon A \to R \otimes R$ commutes with $r^\star$ for all $r \in R$, where $R \otimes R$ is a twisted P.S.H.-module over (R, Id) as in 4.2.2(II).*

Proof. We prove the statement

$$\phi(r^\star(\alpha_{\lambda,\mu})) = r^\star(\phi(\alpha_{\lambda,\mu})) \tag{37}$$

for all $r \in R$ and $\lambda, \mu \in \mathcal{P}$ by induction on λ according to the ordering $<_1$ on $\mathcal{P}$. For $\lambda = (0)$ we easily check that $\alpha_{\lambda,\mu} = \eta_\mu$, so this statement follows from 5.2.1. Otherwise assume that (37) holds for all $\lambda' <_1 \lambda$ and all r, μ. Let l be the first part of λ; it is non-zero and hence $|\lambda^\uparrow| < |\lambda|$.

We first derive the following assertion: $(*)$ for $r_x \in R_x$ with $\deg r_x \leq l$, and partitions ν, μ with $\nu \subseteq \lambda^\uparrow$ we have $\phi(r_x.\alpha_{\nu,\mu}) = r_x.\phi(\alpha_{\nu,\mu})$. We verify this equation by taking inner products with arbitrary irreducible elements $\phi(\alpha_{\lambda',\mu'})$; since ϕ is an isometry this is equivalent to checking whether $\langle r_x.\alpha_{\nu,\mu}, \alpha_{\lambda',\mu'}\rangle = \langle r_x.\phi(\alpha_{\nu,\mu}), \phi(\alpha_{\lambda',\mu'})\rangle$. Now if $\deg r_x = l$ then $r_x \in \mathbf{Z}x_l$, say $r_x = c.x_l$, and if moreover

$\nu = \lambda^\uparrow$ and $(\lambda',\mu') = (\lambda,\mu)$ then by 5.3.2 both sides evaluate to c. In the remaining cases it follows from 5.3.3 and consideration of the degrees that we can have non-zero values at either side of the equation only if $\lambda' <_1 \lambda$. But in that case we may apply the induction hypothesis to obtain $\phi(r^\star_x(\alpha_{\lambda',\mu'})) = r^\star_x(\phi(\alpha_{\lambda',\mu'}))$, from which the desired equality follows by taking inner products with $\phi(\alpha_{\nu,\mu})$.

Now we consider (37). We split this equation into two parts, namely

$$\phi(r^\star(x_l.\alpha_{\lambda^\uparrow,\mu})) = r^\star(\phi(x_l.\alpha_{\lambda^\uparrow,\mu})) \tag{38}$$

and

$$\phi(r^\star(x_l.\alpha_{\lambda^\uparrow,\mu} - \alpha_{\lambda,\mu})) = r^\star(\phi(x_l.\alpha_{\lambda^\uparrow,\mu} - \alpha_{\lambda,\mu})) \tag{39}$$

which together clearly imply (37). By 5.3.3 the term $x_l.\alpha_{\lambda^\uparrow,\mu} - \alpha_{\lambda,\mu}$ only has constituents $\alpha_{\lambda',\mu'}$ with $\lambda' <_1 \lambda$, so the latter equation follows from the induction hypothesis. For the former equation we may compute

$$\phi(r^\star(x_l.\alpha_{\lambda^\uparrow,\mu})) = \phi\Bigl(\sum_{(r)} D(r_1)^*(x_l).r_2^\star(\alpha_{\lambda^\uparrow,\mu})\Bigr) = \sum_{(r)} D(r_1)^*(x_l).\phi(r_2^\star(\alpha_{\lambda^\uparrow,\mu}))$$

(where we applied $(*)$ since $D(r_1)^*(x_l)$ lies in R_x and has degree $\leq l$, and $r_2^\star(\alpha_{\lambda^\uparrow,\mu})$ only has constituents that satisfy the hypothesis of $(*)$; we continue by applying the induction hypothesis:)

$$= \sum_{(r)} D(r_1)^*(x_l).r_2^\star(\phi(\alpha_{\lambda^\uparrow,\mu})) = r^\star(x_l.\phi(\alpha_{\lambda^\uparrow,\mu})) = r^\star(\phi(x_l.\alpha_{\lambda^\uparrow,\mu})),$$

where $(*)$ is again used in the last step; this completes the proof of (38) and of the theorem. □

5.3.5. Corollary. *ϕ is an isomorphism of twisted P.S.H.-modules.*

Proof. The theorem states that ϕ is a morphism of comodules, and because it os also an isomorphism of T-groups, ϕ is a morphism of modules as well, which estasblishes the corollary. □

So we have finally proved the analogue of 5.1.2:

5.3.6. Theorem. *Every elementary twisted P.S.H.-module over (R_E, Id) is isomorphic to the one in 4.2.2(II).* □

§6. Conclusion.

We recapitulate the results we have found. For each of the families $\{\, G_n \mid n \in \mathbf{N}\,\}$ given in §3 of symplectic, unitary or orthogonal groups over a finite field $\mathbf{F}_q$, the graded T-group $A = \bigoplus_{n\in\mathbf{N}} T(G_n)$ with the morphisms ∇_A and Δ_A ("parabolic induction and restriction") is a twisted P.S.H.-module over $(R(q),\tau)$, where $R(q)$ is the analogous graded T-group associated to the family of general linear groups $GL_n(\mathbf{F}_q)$, and τ is its automorphism corresponding to taking transpose inverses. Any twisted P.S.H.-module A over some (R,τ) decomposes as a direct sum $A = \bigoplus_{\alpha\in\mathcal{C}_A} A(\alpha)$ where $\mathcal{C}_A$ is the set of cuspidal elements; all summands $A(\alpha)$ are isomorphic up to a shift in the grading, and the isomorphism class is completely determined by (R,τ), as follows. The P.S.H.-algebra R decomposes as a tensor product $R = \bigotimes_{\rho\in\mathcal{C}^1_R} R(\rho) \otimes \bigotimes_{\rho\in\mathcal{C}^2_R} (R(\rho)\otimes R(\tau(\rho))$ where $\mathcal{C}^i_R$ is a set of representatives of the set of i-element τ-orbits of cuspidal elements in R $(i = 1,2)$. Then $A(\alpha)$ is a corresponding tensor product of tensorands $R_E \otimes R_E$ as in 4.2.2(II) for each $\rho \in \mathcal{C}^1_R$, and tensorands R_E as in 4.2.2(I) for each $\rho \in \mathcal{C}^2_R$—however the grading of each tensorand is scaled by a factor $\deg\rho$, and the grading of the resulting tensor product is shifted by $\deg\alpha$. Therefore the elements $a \in \Omega(A)$ can be parametrised by the following data: (1) an element $\alpha \in \mathcal{C}_A$, (2) a finite (possibly empty) set of elements $\rho_i \in \mathcal{C}^1_R \cup \mathcal{C}^2_R$, with multiplicities $d_i > 0$, and (3) for each $\rho_i \in \mathcal{C}^1_R$ a pair of partitions (λ_i,μ_i) with $|\lambda_i| + |\mu_i| = d_i$, and for each $\rho_i \in \mathcal{C}^2_R$ a partition λ_i with $|\lambda_i| = d_i$. The elements α, ρ_i, d_i form the unique choice for which a is a constituent of $(\prod_i \rho_i^{d_i})\alpha$; in particular we have $\deg\alpha + \sum_i d_i \deg\rho_i = \deg a$.

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